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Birkhoff Averages for Hyperbolic Flows: Variational Principles and Applications

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We establish a higher-dimensional version of multifractal analysis for hyperbolic flows. This means that we consider *simultaneously* the level sets of several Birkhoff averages. Examples are the Lyapunov exponents as well as the pointwise dimension and the local entropy of a given measure. More precisely, we consider multifractal spectra associated to multi-dimensional parameters, obtained by computing the entropy of the level sets associated to the Birkhoff averages. We also consider the more general class of flows with upper semicontinuous metric entropy. The multifractal analysis is obtained here from a variational principle for the topological entropy of the level sets, showing that their topological entropy can be arbitrarily approximated by the entropy of ergodic measures. This principle unifies many results. An analogous principle holds for the Hausdorff dimension. The applications include the study of the regularity of the spectra, the description of how these vary under small perturbations, and the detailed study of the finer structure. The higher-dimensional spectra also exhibit new nontrivial phenomena absent in the one-dimensional multifractal analysis.

KEY WORDS: Variational principles; hyperbolic flows; multifractal analysis.

1. INTRODUCTION

1.1. Motivation

Our work is a contribution to the dimension theory of dynamical systems, and in particular to the associated multifractal analysis. The multifractal spectra—one of the main components of multifractal analysis—encode

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important information about the complexity of the invariant sets of a dynamical system. More precisely, given an invariant function, its associated multifractal spectra describe the complexity of the (invariant) level sets of the function. These functions are usually only measurable and thus their level sets are rarely manifolds. Hence, in order to measure their complexity it is appropriate to use quantities such as the topological entropy or the Hausdorff dimension. Among the invariant functions we can consider several "natural" ones associated to the dynamics, such as Birkhoff averages, Lyapunov exponents, pointwise dimensions, and local entropies.

We briefly recall here the main components of multifractal analysis and in particular the notions of dimension spectrum and more generally of multifractal spectrum of a dynamical system. Let $\Phi = \{\varphi_t\}_{t\in\mathbb{R}}$ be a flow on M preserving a finite measure μ. By Birkhoff's ergodic theorem, for each measurable function $a\colon M \to \mathbb{R}$ with $\int_M |a|\, d\mu < \infty$ the limit

$$a_\Phi(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t a(\varphi_s x)\, ds$$

exists for μ-almost every point $x \in M$. Furthermore, if μ is ergodic (i.e., if every Φ-invariant set has either zero or full measure), then

$$a_\Phi(x) = \frac{1}{\mu(M)}\int_M a\, d\mu \tag{1}$$

for μ-almost every $x \in M$. We note that this does not mean that the identity in (1) is valid for *every* point $x \in M$ for which $a_\Phi(x)$ is well-defined. For each $\alpha \in \mathbb{R}$ we define the level set

$$K_\alpha(a) = \{x \in M : a_\Phi(x) = \alpha\},$$

i.e., the set of points $x \in M$ such that $a_\Phi(x)$ is well-defined and equal to α. We also consider the set

$$K(a) = \left\{ x \in M : \liminf_{t\to\infty} \frac{1}{t}\int_0^t a(\varphi_s x)\, ds < \limsup_{t\to\infty} \frac{1}{t}\int_0^t a(\varphi_s x)\, ds \right\}.$$

Clearly,

$$M = K(a) \cup \bigcup_{\alpha\in\mathbb{R}} K_\alpha(a). \tag{2}$$

We call the decomposition of M in (2) a *multifractal decomposition*. One way to measure the complexity of the sets $K_\alpha(a)$ is to compute their Hausdorff dimension. Namely, we define a function

$$D\colon \{\alpha \in \mathbb{R} : K_\alpha(a) \neq \varnothing\} \to \mathbb{R}$$

by

$$D(\alpha) = \dim_H K_\alpha(a),$$

where $\dim_H A$ denotes the Hausdorff dimension of the set A. The function D is called the *dimension spectrum* of Φ for the Birkhoff averages of a. One can also consider other characteristics to measure the complexity of the sets $K_\alpha(a)$. For example, we obtain the entropy spectra by considering the topological entropy of Φ on $K_\alpha(a)$.

The concept of multifractal analysis was suggested by Halsey, Jensen, Kadanoff, Procaccia, and Shraiman in ref. 10. The first rigorous approach is due to Collet, Lebowitz, and Porzio in ref. 9 for a class of measures invariant under one-dimensional Markov maps. In ref. 13, Lopes considered the measure of maximal entropy for hyperbolic Julia sets, and in ref. 16, Rand studied Gibbs measures for a class of repellers. We refer the reader to the book by Pesin[14] for a detailed discussion and further references.

It is believed by the specialists that the information encoded by the multifractal spectra can be used to recover the dynamical system (possibly up to some appropriate equivalence). This approach is particularly welcome in view of the fact that the multifractal spectra (at least for "natural" invariant functions associated to the dynamics) can be determined with arbitrary precision, while this may not be the case with the dynamical system, that may not be known a priori or at least may not be known with arbitrary precision. We remark that this approach is unrelated to the reconstruction of strange attractors and other characteristics from the analysis of time-series. Namely, instead of dealing with local quantities associated to the behavior of a single trajectory (in terms of an appropriate finite-dimensional coordinate system), we deal here with quantities of global nature, that are encoded in the multifractal spectra.

1.2. Main Results

Our main result is a variational principle for flows with upper semi-continuous metric entropy and potentials with a unique equilibrium measure (see the discussion below on these assumptions). More precisely, we consider multi-dimensional versions of entropy and dimension spectra for flows with upper semi-continuous entropy, and establish a variational principle for these spectra. In particular, we unify and extend the results in the literature. As such, our paper contributes to the global view of the theory of multifractal analysis in the case of flows. We point out that some of our results are new even in the case of one-dimensional multifractal spectra.

We briefly recall here the notion of metric entropy (see Section 2.1 for more details). Let $\Phi = \{\varphi_t\}_{t \in \mathbb{R}}$ be a flow on M and μ a Φ-invariant probability measure. Given a countable partition ξ of M into measurable sets, we define

$$H_\mu(\xi) = -\sum_{C \in \xi} \mu(C) \log \mu(C),$$

with the convention that $0 \log 0 = 0$. The *Kolmogorov–Sinai entropy* of Φ with respect to μ or simply the *metric entropy* of Φ with respect to μ is given by

$$h_\mu(\Phi) = \sup\{h_\mu(\Phi, \xi) \colon H_\mu(\xi) < \infty\},$$

where

$$h_\mu(\Phi, \xi) = \lim_{n \to \infty} \frac{1}{n} H_\mu(\xi_n),$$

and ξ_n is the partition of M into the sets $C_1 \cap C_2 \cap \cdots \cap C_n$ with $C_k \in \varphi_{-k}\xi$ for $k = 0, \ldots, n-1$. For example, the metric entropy of a *hyperbolic* flow, or more precisely a C^1 flow with a hyperbolic set is upper semi-continuous. We recall that given a C^1 flow Φ on a manifold M, a compact Φ-invariant set $X \subset M$ is called *hyperbolic* for the flow Φ if there exist a continuous splitting $T_X M = E^s \oplus E^u \oplus E^0$ and constants $c > 0$ and $\lambda \in (0, 1)$ such that for each $x \in X$ we have:

1. the vector $\frac{d}{dt}(\varphi_t x)|_{t=0}$ generates $E^0(x)$;
2. $d_x\varphi_t E^s(x) = E^s(\varphi_t x)$ and $d_x\varphi_t E^u(x) = E^u(\varphi_t x)$ for each $t \in \mathbb{R}$;
3. $\|d_x\varphi_t v\| \leqslant c\lambda^t \|v\|$ for every $v \in E^s(x)$ and $t > 0$;
4. $\|d_x\varphi_{-t} v\| \leqslant c\lambda^t \|v\|$ for every $v \in E^u(x)$ and $t > 0$.

For example, geodesic flows on compact Riemannian manifolds with negative sectional curvature are hyperbolic. Furthermore, time changes and small C^1 perturbations of hyperbolic flows are also hyperbolic flows. More generally, the metric entropy of an expansive flow is upper semi-continuous.

On the other hand, one can exhibit plenty transformations without a hyperbolic set (and not satisfying specification) for which the metric entropy is still upper semi-continuous. For example, all β-shifts are expansive, and thus the metric entropy is upper semi-continuous (see ref. 12 for details), but for β in a residual set of full Lebesgue measure (although the complement has full Hausdorff dimension) the corresponding β-shift does not satisfy specification (see ref. 18). It follows from work of Walters[(19)] that for every β-shift each Lipschitz function has a unique equilibrium measure. Recall that in the case of topologically mixing hyperbolic flows each Hölder continuous function has a unique equilibrium measure.

The above discussion motivates the formulation of our results not only for flows with a hyperbolic set but more generally for flows with upper semi-continuous metric entropy and potentials with a unique equilibrium measure. Consider now a continuous flow $\Phi = \{\varphi_t\}_{t \in \mathbb{R}}$ on a compact metric space X. Given continuous functions $a_1, a_2\colon X \to \mathbb{R}$ we consider the level sets of Birkhoff averages

$$K_{\alpha_1, \alpha_2} = \left\{ x \in X : \lim_{t \to \infty} \frac{1}{t} \int_0^t a_i(\varphi_s x)\, ds = \alpha_i \text{ for } i = 1, 2 \right\}.$$

We consider the associated *entropy spectrum*

$$\mathscr{E}(\alpha_1, \alpha_2) = h(\Phi \mid K_{\alpha_1, \alpha_2}),$$

where $h(\Phi \mid Z)$ denotes the topological entropy of Φ on the set Z (see Section 2.1 for the definition). We also consider the set

$$\mathscr{P} = \left\{ \left(\int_X a_1\, d\mu, \int_X a_2\, d\mu \right) : \mu \in \mathscr{M} \right\},$$

where $\mathscr{M}$ denotes the family of Φ-invariant probability measures on X. We can now formulate the variational principle for the spectrum $\mathscr{E}$.

Theorem 1. Assume that the metric entropy $\mu \mapsto h_\mu(\Phi)$ is upper semi-continuous, and that for each $c_1, c_2 \in \mathbb{R}$ the function $c_1 a_1 + c_2 a_2$ has a unique equilibrium measure. Then for each $(\alpha_1, \alpha_2) \in \operatorname{int} \mathscr{P}$ we have

$$\mathscr{E}(\alpha_1, \alpha_2) = \max \left\{ h_\mu(\Phi)\colon \left(\int_X a_1\, d\mu, \int_X a_2\, d\mu \right) = (\alpha_1, \alpha_2) \text{ and } \mu \in \mathscr{M} \right\},$$

and there exists an ergodic measure $\mu \in \mathscr{M}$ with $\mu(K_{\alpha_1, \alpha_2}) = 1$ such that

$$h_\mu(\Phi) = \mathscr{E}(\alpha_1, \alpha_2) \qquad \text{and} \qquad \left(\int_X a_1\, d\mu, \int_X a_2\, d\mu \right) = (\alpha_1, \alpha_2).$$

Theorem 1 is a particular case of a much more general result in Theorem 6 later. When Φ is a hyperbolic flow, the statement in Theorem 1 was established by Barreira and Saussol in ref. 4. It also follows from the proof of the theorem that

$$\mathscr{E}(\alpha_1, \alpha_2) = \min\{P_\Phi(q_1 a_1 + q_2 a_2) - q_1\alpha_1 - q_2\alpha_2 : q_1, q_2 \in \mathbb{R}\} \tag{3}$$

for each $(\alpha_1, \alpha_2) \in \operatorname{int} \mathscr{P}$, where P_Φ denotes the topological pressure with respect to the flow Φ (see Section 2.1 for the definition). In particular, the functions $\mathscr{E}$ and $(q_1, q_2) \mapsto P_\Phi(q_1 a_1 + q_2 a_2)$ form a Legendre pair.

1.3. Some Applications

The variational principle in Theorem 1 has several nontrivial applications. Instead of formulating rigorous statements at this point we give instead a brief description of some of the applications. We refer the reader to the main text for rigorous statements.

A first application concerns the description of the so-called finer structure. By considering further continuous functions we can intersect the level sets K_{α_1, α_2} with each of the level sets associated to the Birkhoff averages of the new functions. When this is done we obtain a decomposition of the space X into finer level sets $K \subset K_{\alpha_1, \alpha_2}$. We show in Section 4 that under fairly general assumptions the topological entropy of Φ on K_{α_1, α_2} is carried by a single set K. This is obtained as a consequence of our variational principle. The result should be contrasted with the appearance of "irregular" sets. These are sets of points for which the Birkhoff averages are not defined. While these sets are rather small from the point of view of measure theory they can be as large as desired from the point of view of entropy and dimension (see ref. 14 for a detailed discussion and further references). As such it is not clear a priori whether the remaining "regular" part has a sufficiently large topological entropy when compared to that of K_{α_1, α_2}, i.e., it is not clear whether the topological entropy of Φ on K_{α_1, α_2} is carried by a single set K. It could instead, for example, require a union, perhaps uncountable, of sets of this type to attain the same topological entropy. We refer the reader to the introduction to ref. 6 for a related detailed discussion. The same happens with other global characteristics. This approach conduced in particular to nontrivial applications in number theory related to the study of the distribution of frequencies of digits (see ref. 5).

Another application concerns the study of the regularity of the spectra. In this case we consider hyperbolic flows, and use the fact that for Hölder continuous functions a_1, a_2 the map $(q_1, q_2) \mapsto P_\Phi(q_1 a_1 + q_2 a_2)$ is analytic. We can then use the identity in (3) together with the Implicit function theorem to deduce the analyticity of the spectrum $\mathscr{E}$. The details are given in Section 5.2 where we also consider other multifractal spectra.

Still another application concerns the study of how the multifractal spectra vary under small perturbations. This study intertwines well with the approach described in the beginning and which looks at multifractal spectra as certain multifractal moduli, i.e., as quantities that can be used to recover the dynamics. Namely, it is of interest to understand whether multifractal spectra indeed vary little under small perturbations, since this would tells us that small errors would not affect their experimental study. That this is indeed the case is confirmed in Section 5.3. We also establish explicit formulas that describe how the spectra vary under perturbations.

Our last application concerns the study of dimension spectra in Section 6. We continue to consider hyperbolic flows but study Birkhoff averages taking simultaneously into account the behavior into the future and into the past. Namely, in the hyperbolic set X, we also consider the level sets

$$M_{\beta_1,\beta_2} = \left\{ x \in X : \lim_{t \to -\infty} \frac{1}{t} \int_0^t a_i(\varphi_s x)\, ds = \beta_i \text{ for } i = 1, 2 \right\}.$$

Using the fact that the stable (respectively unstable) local manifold of a given point has exactly the same future Birkhoff average (respectively past Birkhoff average) of that point, the level sets $K_{\alpha_1,\alpha_2} \cap M_{\beta_1,\beta_2}$, that take simultaneously into account the behavior into the future and into the past, can be shown to possess a local product structure (analogous to that of the hyperbolic set X). This observation allows us to use the variational principle to provide a simple description of the associated dimension spectra.

Our study also allows us to exhibit new nontrivial phenomena absent in the one-dimensional multifractal analysis. In particular, while the domain of definition of a one-dimensional spectrum is always an interval, for higher-dimensional spectra the domain may not be convex and may even have empty interior, while still containing an uncountable number of points. Furthermore, the interior of the domain of a higher-dimensional spectrum has in general more than one connected component. We refer to ref. 6 for a related discussion.

Our proofs are based on techniques developed by Barreira, Saussol, and Schmeling in refs. 2, 4, and 6 but require several nontrivial modifications. We require the thermodynamic formalism for flows (see in particular the work by Bowen and Ruelle[8]). An advantage of our approach is that we deal directly with the flows instead of dealing with maps obtained by replacing the flows with certain associated suspension flows, for example by means of Markov systems.

The content of the paper is the following. In Section 2 we recall some basic notions of the thermodynamic formalism for flows. In Section 3 we establish the variational principle. We study the finer structure of the spectra in Section 4. In Section 5 we consider hyperbolic flows, and establish the analyticity of the spectra and study their variation under small perturbations. Section 6 is dedicated to the study of the dimension spectra.

2. THERMODYNAMIC FORMALISM FOR FLOWS

2.1. Topological Pressure and Entropy

Let $\Phi = \{\varphi_t\}_{t\in\mathbb{R}}$ be a continuous flow on a compact metric space (X, d), i.e., a family of transformations $\varphi_t: X \to X$ such that $\varphi_t \circ \varphi_s = \varphi_{t+s}$ for any $t, s \in \mathbb{R}$, and $\varphi_0 x = x$ for any $x \in X$. Given $x \in X$, $t > 0$, and $\varepsilon > 0$, we define

$$B_\varepsilon(x, t) = \{y \in X : d(\varphi_s y, \varphi_s x) < \varepsilon \text{ for any } s \in [0, t]\}.$$

Let $a: X \to \mathbb{R}$ be a continuous function and write

$$a(x, t, \varepsilon) = \sup\left\{\int_0^t a(\varphi_s y)\, ds : y \in B_\varepsilon(x, t)\right\}.$$

For each set $Z \subset X$ and $\alpha \in \mathbb{R}$, we define

$$M(Z, a, \alpha, \varepsilon) = \lim_{T\to\infty} \inf_{\Gamma} \sum_{(x,t)\in\Gamma} \exp(a(x, t, \varepsilon) - \alpha t),$$

where the infimum is taken over all finite or countable sets $\Gamma = \{(x_i, t_i)\}_i$ such that $x_i \in X$ and $t_i \geqslant T$ for each i, and $\bigcup_i B_\varepsilon(x_i, t_i) \supset Z$. Then there exists the limit

$$P_\Phi(a \mid Z) = \lim_{\varepsilon\to 0} \inf\{\alpha: M(Z, a, \alpha, \varepsilon) = 0\}.$$

The number $P_\Phi(a \mid Z)$ is called the *topological pressure of a on Z* (with respect to the flow Φ). We note that the set Z need not be compact nor Φ-invariant (this is crucial in Section 3 since none of sets under consideration will be compact). For simplicity we also write $P_\Phi(a) = P_\Phi(a \mid X)$. We call *topological entropy of Φ on the set Z* to the number $h(\Phi \mid Z) = P_\Phi(0 \mid Z)$.

We now consider the set $\mathcal{M}_\Phi(X)$ of Φ-invariant probability measures on X. Recall that a measure μ on X is *Φ-invariant* if $\mu(\varphi_t A) = \mu(A)$ for any $t \in \mathbb{R}$ and any set $A \subset X$. With the weak $*$ topology the space $\mathcal{M}_\Phi(X)$ is compact and metrizable. Recall also that a measure μ on X is *ergodic* if for any Φ-invariant set $A \subset X$ (i.e., any set such that $\varphi_t A = A$ for every $t \in \mathbb{R}$) we have $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

For each measure $\mu \in \mathscr{M}_\Phi(X)$ there exists the limit

$$h_\mu(\Phi) = \lim_{\varepsilon \to 0} \inf\{h(Z, \varepsilon) : \mu(Z) = 1\}, \tag{4}$$

where

$$h(Z, \varepsilon) = \inf\{\alpha : M(Z, 0, \alpha, \varepsilon) = 0\}. \tag{5}$$

Proposition 2. If Φ is a continuous flow on a compact metric space and $\mu \in \mathscr{M}_\Phi(X)$ is ergodic, then $h_\mu(\Phi)$ is the entropy of Φ with respect to μ, i.e., the entropy of φ_1 with respect to μ.

In the case of ergodic measures, we can thus use (4) and (5) to compute the entropy. An analogous result was established by Pesin in ref. 14, Theorem 11.6 in the discrete-time case. The proof of Proposition 2 is a simple modification of the proof of that statement and hence it is not reproduced here.

2.2. BS-Dimension

We now recall a Carathéodory characteristic introduced by Barreira and Saussol in ref. 3. It is a generalization of the topological entropy, and is a version of the Carathéodory characteristic introduced by Barreira and Schmeling in ref. 7 in the discrete-time case.

Let Φ be a continuous flow on a compact metric space X and $u: X \to \mathbb{R}$ a positive continuous function. For each set $Z \subset X$ and $\alpha \in \mathbb{R}$, we define

$$N(Z, u, \alpha, \varepsilon) = \lim_{T \to \infty} \inf_{\Gamma} \sum_{(x,t) \in \Gamma} \exp(-\alpha u(x, t, \varepsilon)),$$

where the infimum is taken over all finite or countable sets $\Gamma = \{(x_i, t_i)\}_i$ such that $x_i \in X$ and $t_i \geqslant T$ for each i, and $\bigcup_i B_\varepsilon(x_i, t_i) \supset Z$. Setting

$$\dim_{u,\varepsilon} Z = \inf\{\alpha : N(Z, u, \alpha, \varepsilon) = 0\},$$

there exists the limit

$$\dim_u Z = \lim_{\varepsilon \to 0} \dim_{u,\varepsilon} Z.$$

Following Pesin,[14] the number $\dim_u Z$ is called the *BS-dimension of* Z *(with respect to* u*)*. When $u=1$ we have $\dim_u Z = h(\Phi \mid Z)$.

It follows easily from the definitions that the topological pressure and the BS-dimension are related in the following manner.

Proposition 3. The unique root of $P_\Phi(-\alpha u \mid Z) = 0$ is $\alpha = \dim_u Z$.

For each probability Borel measure μ on X, let

$$\dim_{u,\varepsilon} \mu = \inf\{\dim_{u,\varepsilon} Z \colon \mu(Z) = 1\}.$$

Then there exists the limit

$$\dim_u \mu = \lim_{\varepsilon \to 0} \dim_{u,\varepsilon} \mu.$$

The number $\dim_u \mu$ is called the *BS-dimension of* μ *(with respect to* u*)*. For each ergodic measure $\mu \in \mathcal{M}_\Phi(X)$ we have $\dim_u \mu = h_\mu(\Phi) / \int_X u \, d\mu$. The proof of this identity can be obtained in an analogous manner to that in the discrete-time case in ref. 7.

2.3. Properties of the Pressure

We recall here some of the basic properties of the topological pressure. See refs. 12, 17, and 20 for full details.

Proposition 4. If Φ is a continuous flow on a compact metric space X and $a \colon X \to \mathbb{R}$ is a continuous function, then

$$P_\Phi(a) = \sup\left\{ h_\mu(\Phi) + \int_X a \, d\mu : \mu \in \mathcal{M}_\Phi(X) \right\}. \tag{6}$$

A measure $\mu \in \mathcal{M}_\Phi(X)$ is called an *equilibrium measure* for the function a (with respect to the flow Φ) if the supremum in (6) is attained at this measure, i.e., if $P_\Phi(a) = h_\mu(\Phi) + \int_X a \, d\mu$. We denote by $C(X)$ the space of continuous functions $a \colon X \to \mathbb{R}$ equipped with the supremum norm and by $D(X) \subset C(X)$ the family of continuous functions with a unique equilibrium measure. For a finite set $K \subset C(X)$ we denote by $\operatorname{span} K \subset C(X)$ the linear space generated by the functions in K.

Proposition 5. If Φ is a continuous flow on a compact metric space X such that $\mu \mapsto h_\mu(\Phi)$ is upper semi-continuous, then:

1. each $a \in C(X)$ has equilibrium measures, and $D(X)$ is dense in $C(X)$;

2. given $a, b \in C(X)$, the map $\mathbb{R} \ni t \mapsto P_\Phi(a+tb)$ is differentiable at $t=0$ if and only if $a \in D(X)$, in which case the unique equilibrium measure μ_a of a is ergodic and satisfies

$$\frac{d}{dt} P_\Phi(a+tb)|_{t=0} = \int_X b\, d\mu_a; \tag{7}$$

3. if $\operatorname{span}\{a, b\} \subset D(X)$ then the function $t \mapsto P_\Phi(a+tb)$ is of class C^1.

A flow Φ on a metric space X is *expansive* if there exists $\varepsilon > 0$ such that given $x, y \in X$ and a continuous function $s: \mathbb{R} \to \mathbb{R}$ with $s(0)=0$ satisfying

$$d(\varphi_t x, \varphi_{s(t)} x) < \varepsilon \qquad \text{and} \qquad d(\varphi_t x, \varphi_{s(t)}\, y) < \varepsilon$$

for every $t \in \mathbb{R}$, we must have $x = y$. If Φ is an expansive flow then the entropy is upper semi-continuous (see ref. 20).

We say that a function $a: X \to \mathbb{R}$ is *Φ-cohomologous* to a function $b: X \to \mathbb{R}$ if there exists a bounded measurable function $q: X \to \mathbb{R}$ such that

$$a(x) - b(x) = \lim_{t \to 0} \frac{q(\varphi_t x) - q(x)}{t}.$$

In this case $P_\Phi(a \mid X) = P_\Phi(b \mid X)$.

3. MULTIFRACTAL ANALYSIS

We introduce in this section a multifractal spectrum for ratios of Birkhoff averages of a given flow. We then establish a variational principle for this spectrum.

3.1. Conditional Variational Principle

Let again $\Phi = \{\varphi_t\}_{t \in \mathbb{R}}$ be a continuous flow on the compact metric space X. We consider $d \in \mathbb{N}$ and vectors $(A, B) \in C(X)^d \times C(X)^d$,

$$A = (a_1, \ldots, a_d) \qquad \text{and} \qquad B = (b_1, \ldots, b_d),$$

with $B > 0$ (i.e., with $b_i > 0$ for $i = 1, \ldots, d$). We equip $\mathbb{R}^d$ with the norm $\|\alpha\| = |\alpha_1| + \cdots + |\alpha_d|$ and $C(X)^d$ with the corresponding induced norm.

Given $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ we define

$$K_\alpha = K_\alpha(A, B) = \bigcap_{i=1}^{d} \left\{ x \in X : \lim_{t \to \infty} \frac{\int_0^t a_i(\varphi_s x)\, ds}{\int_0^t b_i(\varphi_s x)\, ds} = \alpha_i \right\}. \tag{8}$$

Let $u: X \to \mathbb{R}$ be a positive continuous function. We define the *BS-dimension spectrum* $\mathscr{F}_u: \mathbb{R}^d \to \mathbb{R}$ *of* (A, B) (with respect to u and Φ) by

$$\mathscr{F}_u(\alpha) = \dim_u K_\alpha(A, B).$$

We also consider the function $\mathscr{P} = \mathscr{P}_{A,B}: \mathscr{M}_\Phi(X) \to \mathbb{R}^d$ given by

$$\mathscr{P}_{A,B}(\mu) = \left(\frac{\int_X a_1\, d\mu}{\int_X b_1\, d\mu}, \ldots, \frac{\int_X a_d\, d\mu}{\int_X b_d\, d\mu} \right). \tag{9}$$

Given $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ and $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{R}^d$ we write

$$\alpha * \beta = (\alpha_1 \beta_1, \ldots, \alpha_d \beta_d) \in \mathbb{R}^d \qquad \text{and} \qquad \langle \alpha, \beta \rangle = \sum_{i=1}^{d} \alpha_i \beta_i \in \mathbb{R}.$$

Theorem 6. Let Φ be a continuous flow on a compact metric space X such that $\mu \mapsto h_\mu(\Phi)$ is upper semi-continuous, and consider functions $(A, B) \in C(X)^d \times C(X)^d$ with $B > 0$ and $u \in C(X)$ with $u > 0$ such that

$$\operatorname{span}\{a_1, b_1, \ldots, a_d, b_d, u\} \subset D(X).$$

If $\alpha \in \operatorname{int} \mathscr{P}(\mathscr{M}_\Phi(X))$ then $K_\alpha \neq \varnothing$ and the following properties hold:

1. $\mathscr{F}_u(\alpha)$ satisfies the variational principle

$$\mathscr{F}_u(\alpha) = \max \left\{ \frac{h_\mu(\Phi)}{\int_X u\, d\mu} : \mu \in \mathscr{M}_\Phi(X) \text{ and } \mathscr{P}(\mu) = \alpha \right\}; \tag{10}$$

2. $\mathscr{F}_u(\alpha) = \min\{T_u(\alpha, q) : q \in \mathbb{R}^d\}$, where $T_u(\alpha, q)$ is the unique number satisfying

$$P_\Phi(\langle q, A - \alpha * B \rangle - T_u(\alpha, q)\, u) = 0; \tag{11}$$

3. there exists an ergodic measure $\mu_\alpha \in \mathscr{M}_\Phi(X)$ such that $\mathscr{P}(\mu_\alpha) = \alpha$, $\mu_\alpha(K_\alpha) = 1$, and $\dim_u \mu_\alpha = \mathscr{F}_u(\alpha)$.

Furthermore, if $\alpha \notin \mathscr{P}(\mathscr{M}_\Phi(X))$ then $K_\alpha = \varnothing$.

The proof of Theorem 6 is given in Section 3.2. When Φ is a hyperbolic flow, the statement in Theorem 6 was established by Barreira and Saussol in ref. 4 in the case of the entropy (i.e., when $u=1$). More precisely, the authors include a complete proof when $d=1$ and describe the required changes for an arbitrary d.

It also follows from the proof of Theorem 6 that μ_α can be chosen to be an equilibrium measure of the function $\langle q(\alpha), A-\alpha * B\rangle - \mathscr{F}_u(\alpha)\, u$, where $q(\alpha) \in \mathbb{R}^d$ is any number such that

$$P_\Phi(\langle q(\alpha), A-\alpha * B\rangle - \mathscr{F}_u(\alpha)\, u) = 0.$$

We note that $q(\alpha)$ is not necessarily unique and thus μ_α may also not be unique. The function T_u is implicitly defined by (11) and thus, by Proposition 5, the function $(p, \alpha, q) \mapsto P_\Phi(\langle q, A-\alpha * B\rangle - pu)$ is of class C^1. Furthermore,

$$\frac{\partial}{\partial p} P_\Phi(\langle q, A-\alpha * B\rangle - pu)\bigg|_{(p,q)=(T_u(\alpha,q),q)} = -\int_X u\, d\mu_q < 0,$$

where μ_q is the equilibrium measure of $\langle q, A-\alpha * B\rangle - T_u(\alpha, q)\, u$. It then follows from the Implicit function theorem that T_u is of class C^1 in $\mathbb{R}^d \times \mathbb{R}^d$. Hence, for each α the minimum in statement 2 of Theorem 6 is attained at a point $q \in \mathbb{R}^d$ such that $\partial_q T_u(\alpha, q) = 0$.

We now consider the particular case when there is a cohomology relation between u and B. Namely, let $\gamma = (\gamma_1, \ldots, \gamma_d) \in \mathbb{R}^d$ be such that b_i is Φ-cohomologous to $\gamma_i u$ for $i = 1, \ldots, d$. Then the number $T_u(\alpha, q)$ satisfying (11) is equivalently defined by requiring that

$$P_\Phi(\langle q, A\rangle - [\langle q, \alpha * \gamma\rangle + T_u(\alpha, q)]\, u) = 0. \tag{12}$$

By statement 2 of Theorem 6 we obtain

$$\mathscr{F}_u(\alpha) = \min\{Q_u(q) - \langle q, \alpha * \gamma\rangle : q \in \mathbb{R}^d\},$$

where $Q_u(q) = \langle q, \alpha * \gamma\rangle + T_u(\alpha, q)$. By (12) the function Q_u is well-defined (the right-hand side does not depend on α). Since the minimum is obtained at a point $q \in \mathbb{R}^d$ satisfying $\partial_q T_u(\alpha, q) = 0$, setting $\gamma^{-1} = (\gamma_1^{-1}, \ldots, \gamma_d^{-1})$ we obtain $\alpha = \gamma^{-1} * \partial_q Q_u(q)$.

3.2. Proof of Theorem 6

The proof follows arguments of Barreira, Saussol, and Schmeling in ref. 6 in the context of discrete-time. For simplicity we use the notation $\mu(\psi) = \int_X \psi\, d\mu$.

Consider $\alpha \in \mathbb{R}^d$ such that $K_\alpha \neq \varnothing$. We choose $x \in K_\alpha$ and define a sequence $(\mu_n)_{n \in \mathbb{N}}$ of probability measures on X by $\mu_n(a) = \frac{1}{n} \int_0^n a(\varphi_s x)\, ds$ for every $a \in C(X)$. Since $\mathscr{M}_\Phi(X)$ is compact, this sequence has at least one accumulation point $\mu \in \mathscr{M}_\Phi(X)$ and

$$\begin{aligned}\alpha &= \left(\lim_{t \to +\infty} \frac{\int_0^t a_1(\varphi_s x)\, ds}{\int_0^t b_1(\varphi_s x)\, ds}, \ldots, \lim_{t \to +\infty} \frac{\int_0^t a_d(\varphi_s x)\, ds}{\int_0^t b_d(\varphi_s x)\, ds} \right) \\ &= \left(\lim_{n \to +\infty} \frac{\mu_n(a_1)}{\mu_n(b_1)}, \ldots, \lim_{n \to +\infty} \frac{\mu_n(a_d)}{\mu_n(b_d)} \right) \\ &= \left(\frac{\mu(a_1)}{\mu(b_1)}, \ldots, \frac{\mu(a_d)}{\mu(b_d)} \right) = \mathscr{P}(\mu) \in \mathscr{P}(\mathscr{M}_\Phi(X)).\end{aligned}$$

Let now $\alpha \in \operatorname{int} \mathscr{P}(\mathscr{M}_\Phi(X))$. The existence of the maximum in (10) is a consequence of the upper semi-continuity of $\mu \mapsto h_\mu(\Phi)/\int_X u\, d\mu$, the compactness of $\mathscr{M}_\Phi(X)$, and the continuity of $\mathscr{P}$. For each $q \in \mathbb{R}^d$ we write

$$\varphi_{q,\alpha} = \langle q, A - \alpha * B \rangle - \mathscr{F}_u(\alpha)\, u \qquad \text{and} \qquad F_\alpha(q) = P_\Phi(\varphi_{q,\alpha}).$$

Let $r > 0$ be the distance from α to $\mathbb{R}^d \setminus \mathscr{P}(\mathscr{M}_\Phi(X))$ and choose q such that

$$\|q\| \geqslant \frac{\dim_u X \cdot \sup u + F_\alpha(0)}{r \min_i \inf b_i} = R.$$

If $\lambda \in (0, 1)$ and $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{R}^d$ with $\beta_i = \alpha_i + \frac{1}{d} \lambda r \operatorname{sgn} q_i$, then

$$\|\beta - \alpha\| = \sum_{i=1}^d |\beta_i - \alpha_i| = \sum_{i=1}^d \frac{1}{d} \lambda r \operatorname{sgn} |q_i| = \lambda r < r.$$

Hence $\beta \in \mathscr{P}(\mathscr{M}_\Phi(X))$ and there is $\mu \in \mathscr{M}_\Phi(X)$ such that $\mu(A - \beta * B) = 0$. We obtain

$$\begin{aligned}\langle q, \mu(A - \alpha * B) \rangle &= \langle q, \mu((\beta - \alpha) * B) \rangle \\ &= \sum_{i=1}^d q_i\, \mu((\beta_i - \alpha_i) * b_i) = \sum_{i=1}^d \frac{1}{d} \lambda r q_i \operatorname{sgn} q_i \int_X b_i\, d\mu \\ &= \frac{1}{d} \lambda r \sum_{i=1}^d |q_i| \int_X b_i\, d\mu \geqslant \lambda r \|q\| \min_i \inf b_i.\end{aligned}$$

By Proposition 4 and since $h_\mu(\Phi) \geqslant 0$ we have

$$\begin{aligned} F_\alpha(q) &\geqslant h_\mu(\Phi) + \mu(\varphi_{q,\alpha}) = h_\mu(\Phi) + \langle q, \mu(A - \alpha * B)\rangle - \mathscr{F}_u(\alpha)\,\mu(u) \\ &\geqslant \|q\|\,\lambda r \min_i \inf b_i - \dim_u X \cdot \sup u \\ &\geqslant \lambda[\dim_u X \cdot \sup u + F_\alpha(0)] - \dim_u X \cdot \sup u. \end{aligned}$$

Since $\lambda \in (0, 1)$ is arbitrary, taking $\lambda \to 1$ we conclude that $F_\alpha(q) \geqslant F_\alpha(0)$ for every $q \in \mathbb{R}^d$ such that $\|q\| \geqslant R$. By Proposition 5 the function F is of class C^1 and hence it reaches a minimum at a point $q = q(\alpha)$ with $\|q(\alpha)\| \leqslant R$, satisfying $\partial_q F_\alpha(q(\alpha)) = 0$. By (7),

$$\mu_\alpha(A - \alpha * B) = \partial_q F_\alpha(q(\alpha)) = 0,$$

where μ_α is the equilibrium measure of $\varphi_{q,\alpha}$. Thus, $\mathscr{P}(\mu_\alpha) = \alpha$. Moreover,

$$F_\alpha(q(\alpha)) = h_{\mu_\alpha}(\Phi) - \mathscr{F}_u(\alpha) \int_X u\, d\mu_\alpha. \tag{13}$$

Let now $x \in K_\alpha$. For $i = 1,..., d$, we have

$$\lim_{t \to \infty} \frac{\int_0^t a_i(\varphi_s x)\, ds}{\int_0^t b_i(\varphi_s x)\, ds} = \alpha_i.$$

Since $b_i > 0$, taking $\delta > 0$ there exists τ such that, for all $t > \tau$,

$$\left| \frac{\int_0^t a_i(\varphi_s x)\, ds}{\int_0^t b_i(\varphi_s x)\, ds} - \alpha_i \right| < \frac{\delta}{dM},$$

where $M = \max_{i \in \{1,..., d\}} \max_{x \in X} b_i(x)$. We define $A_t(x) = \int_0^t A(\varphi_s x)\, ds$ and $B_t(x) = \int_0^t B(\varphi_s x)\, ds$, and let

$$L_{\delta,\tau} = \{x \in X : \|A_t(x) - \alpha * B_t(x)\| < \delta t \text{ for all } t \geqslant \tau\}.$$

We obtain

$$\begin{aligned} \|A_t(x) - \alpha * B_t(x)\| &= \sum_{i=1}^d \left| \int_0^t a_i(\varphi_s x)\, ds - \alpha_i \int_0^t b_i(\varphi_s x)\, ds \right| \\ &< \frac{\delta}{dM} \sum_{i=1}^d \int_0^t b_i(\varphi_s x)\, ds < \delta t, \end{aligned}$$

and hence $x \in L_{\delta,\tau} \subseteq \bigcup_{\tau \in \mathbb{R}} L_{\delta,\tau}$ for any $\delta > 0$. Thus $K_\alpha \subseteq \bigcap_{\delta > 0} \bigcup_{\tau \in \mathbb{R}} L_{\delta,\tau}$.

Since X is compact each function a_i is uniformly continuous. Hence, there exists $\varepsilon > 0$ such that, whenever $(x, t) \in X \times [0, \infty)$, if $y, z \in B_\varepsilon(x, t)$ (and, thus $d(\varphi_s y, \varphi_s z) < 2\varepsilon$) then $|a_i(\varphi_s z) - a_i(\varphi_s y)| < \delta/d$, for all $0 \leqslant s \leqslant t$. Consider

$$A(x, t, \varepsilon) = (a_1(x, t, \varepsilon), \dots, a_d(x, t, \varepsilon))$$

and take $y \in B_\varepsilon(x, t)$. We obtain

$$\begin{aligned}
\|A(x, t, \varepsilon) - A_t(y)\| &= \sum_{i=1}^{d} \left| a_i(x, t, \varepsilon) - \int_0^t a_i(\varphi_s y)\, ds \right| \\
&\leqslant d \sup \left\{ \int_0^t |a_i(\varphi_s z) - a_i(\varphi_s y)|\, ds : z \in B_\varepsilon(x, t) \right\} \\
&\leqslant d \sup \left\{ \int_0^t \frac{\delta}{d}\, ds : z \in B_\varepsilon(x, t) \right\} \leqslant \delta t,
\end{aligned}$$

and analogously, $\|B(x, t, \varepsilon) - B_t(y)\| \leqslant \delta t$.

Take $q \in \mathbb{R}^d$. Given $(x, t) \in X \times [\tau, \infty)$ such that $B_\varepsilon(x, t) \cap L_{\delta, \tau} \neq \varnothing$ and $y \in B_\varepsilon(x, t) \cap L_{\delta, \tau}$, we have

$$\begin{aligned}
&-\langle q, A - \alpha * B \rangle(x, t, \varepsilon) \\
&\quad \leqslant |\langle q, A - \alpha * B \rangle(x, t, \varepsilon)| \\
&\quad \leqslant \|q\| \cdot \|A(x, t, \varepsilon) - \alpha * B(x, t, \varepsilon)\| \\
&\quad \leqslant \|q\| \cdot \|A(x, t, \varepsilon) - A_t(y)\| + \|q\| \cdot \|\alpha * B_t(y) - \alpha * B(x, t, \varepsilon)\| \\
&\qquad + \|q\| \cdot \|A_t(y) - \alpha * B_t(y)\| \\
&\quad \leqslant \|q\|\, (\delta t + \|\alpha\|\, \delta t + \delta t) = c\, \delta t,
\end{aligned}$$

where $c = (2 + \|\alpha\|)\, \|q\|$. Hence,

$$\begin{aligned}
\exp(-\mathscr{F}_u(\alpha)\, u(x, t, \varepsilon) - \beta t) &= \exp(\varphi_{q, \alpha}(x, t, \varepsilon) - \langle q, A - \alpha * B \rangle(x, t, \varepsilon) - \beta t) \\
&\leqslant \exp(\varphi_{q, \alpha}(x, t, \varepsilon) - (\beta - c\delta)\, t)
\end{aligned}$$

for all $\beta \in \mathbb{R}$. Let $T \geqslant \tau$ and consider a finite or countable family $\Gamma = \{(x_i, t_i)\}_i$ where $x_i \in X$ and $t_i \geqslant T$ for each i, $L_{\delta, \tau} \subset \bigcup_i B_\varepsilon(x_i, t_i)$, and there exists no (x_i, t_i) such that $B_\varepsilon(x_i, t_i) \cap L_{\delta, \tau} = \varnothing$. Then

$$\sum_{(x, t) \in \Gamma} \exp(-\mathscr{F}_u(\alpha)\, u(x, t, \varepsilon) - \beta t) \leqslant \sum_{(x, t) \in \Gamma} \exp(\varphi_{q, \alpha}(x, t, \varepsilon) - (\beta - c\delta)\, t).$$

Taking the infimum over Γ, and letting $T \to \infty$ we obtain

$$M(L_{\delta,\tau}, -\mathscr{F}_u(\alpha)\, u, \beta, \varepsilon) \leqslant M(L_{\delta,\tau}, \varphi_{q,\alpha}, \beta - c\delta, \varepsilon).$$

Letting $\varepsilon \to 0$ yields $P_\Phi(-\mathscr{F}_u(\alpha)\, u \mid L_{\delta,\tau}) \leqslant P_\Phi(\varphi_{q,\alpha} \mid L_{\delta,\tau}) + c\delta$ for every $\delta > 0$ and $q \in \mathbb{R}^d$. By Proposition 3 and the properties of the topological pressure,

$$\begin{aligned} 0 = P_\Phi(-\mathscr{F}_u(\alpha)\, u \mid K_\alpha) &\leqslant P_\Phi\left(-\mathscr{F}_u(\alpha)\, u \,\middle|\, \bigcup_{\tau \in \mathbb{R}} L_{\delta,\tau}\right) \\ &= \sup_{\tau > 0} P_\Phi(-\mathscr{F}_u(\alpha)\, u \mid L_{\delta,\tau}) \leqslant P_\Phi(\varphi_{q,\alpha} \mid L_{\delta,\tau}) + c\delta \leqslant F_\alpha(q) + c\delta \end{aligned}$$

for every $\delta > 0$ and $q \in \mathbb{R}^d$. Since δ is arbitrary, we obtain $F_\alpha(q) \geqslant 0$. By Proposition 5 and (13), the measure μ_α is ergodic and

$$\dim_u \mu_\alpha = \frac{h_{\mu_\alpha}(\Phi)}{\int_X u\, d\mu_\alpha} \geqslant \mathscr{F}_u(\alpha).$$

Since $\mu_\alpha(A - \alpha * B) = 0$, Birkhoff's ergodic theorem shows that $\mu_\alpha(K_\alpha) = 1$. This implies that

$$\mathscr{F}_u(\alpha) = \dim_u K_\alpha = \lim_{\varepsilon \to 0} \dim_{u,\varepsilon} K_\alpha \geqslant \lim_{\varepsilon \to 0} \dim_{u,\varepsilon} \mu_\alpha = \dim_u \mu_\alpha,$$

and hence $\dim_u \mu_\alpha = \mathscr{F}_u(\alpha)$. Therefore

$$\begin{aligned} \min\{F_\alpha(q) \colon q \in \mathbb{R}^d\} &= F_\alpha(q(\alpha)) = h_{\mu_\alpha}(\Phi) - \mathscr{F}_u(\alpha) \int_X u\, d\mu_\alpha \\ &= h_{\mu_\alpha}(\Phi) - \frac{h_{\mu_\alpha}(\Phi)}{\int_X u\, d\mu_\alpha} \int_X u\, d\mu_\alpha = 0. \end{aligned}$$

Take $\mu \in \mathscr{M}_\Phi(X)$ such that $\mathscr{P}(\mu) = \alpha$. Then $\mu(\langle q, A - \alpha * B \rangle) = 0$ and by Proposition 4,

$$\begin{aligned} 0 = \min\{F_\alpha(q) \colon q \in \mathbb{R}^d\} &\geqslant \inf_{q \in \mathbb{R}^d} \{h_\mu(\Phi) + \mu(\langle q, A - \alpha * B \rangle - \mathscr{F}_u(\alpha)\, u)\} \\ &\geqslant \inf_{q \in \mathbb{R}^d} \left\{h_\mu(\Phi) - \mathscr{F}_u(\alpha) \int_X u\, d\mu\right\} = h_\mu(\Phi) - \mathscr{F}_u(\alpha) \int_X u\, d\mu. \end{aligned}$$

Therefore $h_\mu(\Phi) / \int_X u\, d\mu \leqslant \mathscr{F}_u(\alpha)$, with equality when $\mu = \mu_\alpha$. This establishes statements 1 and 3 of the theorem.

Furthermore, since $F_\alpha(q(\alpha)) = 0$,

$$\mathscr{F}_u(\alpha) = T_u(\alpha, q(\alpha)) \geqslant \inf\{T_u(\alpha, q) : q \in \mathbb{R}^d\}.$$

On the other hand,

$$F_\alpha(q) \geqslant 0 = P_\Phi(\langle q, A - \alpha * B \rangle - T_u(\alpha, q)\, u),$$

and hence $\mathscr{F}_u(\alpha) \leqslant \inf\{T_u(\alpha, q) : q \in \mathbb{R}^d\}$. This establishes statement 2 and the proof of the theorem is completed.

4. FINER STRUCTURE

We now study in even greater detail the structure of the level sets K_α (see (8)). A related study was effected by Barreira, Saussol, and Schmeling in ref. 6 in the discrete-time case.

Let Φ be a continuous flow on a compact metric space X. Let $d \in \mathbb{N}$ and $A, B \in C(X)^d$ (we no longer assume that $B > 0$). Let also $u \in C(X)$ be a positive function, and consider the vector $U = (u, \dots, u) \in C(X)^d$. We define $a_t(x) = \int_0^t a(\varphi_s x)\, ds$ for each $a \in C(X)$, and let $F: \mathscr{D} \subset \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be a continuous function. We assume that

$$\left(\frac{A_t(x)}{u_t(x)}, \frac{B_t(x)}{u_t(x)} \right) \in \mathscr{D} \tag{14}$$

for every $t \in \mathbb{R}$ and $x \in X$. We also consider the sets

$$L_\alpha^F = \left\{ x \in X : \lim_{t \to \infty} F\left(\frac{A_t(x)}{u_t(x)}, \frac{B_t(x)}{u_t(x)} \right) = \alpha \right\}$$

for each $\alpha \in \mathbb{R}^d$. When $F(X, Y) = X * Y^{-1}$ this coincides with the set K_α in (8). We also consider the associated multifractal spectrum $\mathscr{G}_u$ defined by $\mathscr{G}_u(\alpha) = \dim_u L_\alpha^F$ for each $\alpha \in \mathbb{R}^d$.

We want to establish a relation between the BS-dimension of a given set L_α^F (with respect to u) and the BS-dimension of the sets

$$K_{\beta,\gamma} = \left\{ x \in X : \lim_{t \to \infty} \left(\frac{A_t(x)}{u_t(x)}, \frac{B_t(x)}{u_t(x)} \right) = (\beta, \gamma) \right\},$$

with $\beta, \gamma \in \mathbb{R}^d$. These new sets are of the same type of those in (8) but now corresponding to a parameter (β, γ) of dimension $2d$ and to vectors of functions (A, B), $(U, U) \in C(X)^{2d}$. We write

$$\mathscr{H}_u(\beta, \gamma) = \dim_u K_{\beta,\gamma}.$$

Given $q \in \mathbb{R}^{2d}$, let $S_u(q)$ be the unique number satisfying

$$P_\Phi(\langle q, (A, B)\rangle - S_u(q)\, u) = 0$$

and let μ_q be an equilibrium measure of $\langle q, (A, B)\rangle - S_u(q)\, u$ (this measure will be unique in our context).

By Theorem 6 applied to the $2d$-dimensional spectrum $\mathscr{H}_u$ (see also the discussion after Theorem 6), we obtain the following.

Theorem 7. Let Φ be a continuous flow on a compact metric space X such that $\mu \mapsto h_\mu(\Phi)$ is upper semi-continuous. If $\operatorname{span}\{a_1, b_1, ..., a_d, b_d, u\} \subset D(X)$, then

$$\mathscr{H}_u(\partial_q S_u(q)) = S_u(q) - \langle q, \partial_q S_u(q)\rangle$$

and $\mu_q(K_{\partial_q S_u(q)}) = 1$ for every $q \in \mathbb{R}^{2d}$.

Observe that $\mathscr{H}_u$ and S_u form a Legendre pair.

Proposition 8. Let Φ be a continuous flow on a compact metric space X and $F\colon \mathscr{D} \subset \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ a continuous function such that (14) holds for every $t \in \mathbb{R}$ and $x \in X$. Then

$$\mathscr{G}_u(\alpha) \geqslant \sup\{\mathscr{H}_u(\beta, \gamma)\colon (\beta, \gamma) \in F^{-1}(\alpha)\} \tag{15}$$

for every $\alpha \in F(\mathscr{D})$.

Proof. Take $(\beta, \gamma) \in \mathscr{D}$ and $x \in K_{\beta, \gamma}$. By the continuity of F,

$$\lim_{t \to \infty} F\left(\frac{A_t(x)}{u_t(x)}, \frac{B_t(x)}{u_t(x)}\right) = F\left(\lim_{t \to \infty} \frac{A_t(x)}{u_t(x)}, \lim_{t \to \infty} \frac{B_t(x)}{u_t(x)}\right) = F(\beta, \gamma).$$

This implies that

$$\bigcup_{(\beta, \gamma) \in F^{-1}(\alpha)} K_{\beta, \gamma} \subseteq L^F_\alpha.$$

Since $K_{\beta, \gamma} \subset L^F_\alpha$ for every $(\beta, \gamma) \in F^{-1}(\alpha)$, we have $\dim_u K_{\beta, \gamma} \leqslant \dim_u L^F_\alpha$. This completes the proof. ∎

When F is chosen in such a way that L^F_α is of the form $K_\alpha(G, H)$ for some vectors $G, H \in C(X)^d$, we can apply Theorem 6 and explicitly determine a point $(\beta, \gamma) \in F^{-1}(\alpha)$ such that $\mathscr{G}_u(\alpha) = \mathscr{H}_u(\beta, \gamma)$.

Example 1. Consider vectors $A,\ B \in C(X)^d$ and constants $\gamma_i \in \mathbb{R}^d$ for $i \in \{1,\ldots,6\}$, such that $\gamma_4 * A(x) + \gamma_5 * B(x) + \gamma_6 \neq 0$ for every $x \in X$. We define a function $F\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ by

$$F(X, Y) = (\gamma_1 * X + \gamma_2 * Y + \gamma_3) * (\gamma_4 * X + \gamma_5 * Y + \gamma_6)^{-1}.$$

Since

$$F\left(\frac{A_t(x)}{u_t(x)}, \frac{B_t(x)}{u_t(x)}\right) = (\gamma_1 * A + \gamma_2 * B + \gamma_3)_t\,(x) * (\gamma_4 * A + \gamma_5 * B + \gamma_6)_t^{-1}\,(x)$$

for every $t \in \mathbb{R}$ and $x \in X$, we have $L_\alpha^F = K_\alpha(G, H)$, where

$$G = \gamma_1 * A + \gamma_2 * B + \gamma_3 = (g_1,\ldots, g_d),$$

$$H = \gamma_4 * A + \gamma_5 * B + \gamma_6 = (h_1,\ldots, h_d).$$

Hence, defining $\mathscr{P} = \mathscr{P}_{G,H}\colon \mathscr{M}_\Phi(X) \to \mathbb{R}^d$ by

$$\mathscr{P}_{G,H}(\mu) = \left(\frac{\int_X g_1\, d\mu}{\int_X h_1\, d\mu},\ldots, \frac{\int_X g_d\, d\mu}{\int_X h_d\, d\mu}\right),$$

it follows from Theorem 6 that for each $\alpha \in \operatorname{int} \mathscr{P}(\mathscr{M}_\Phi(X))$ there exists an ergodic measure μ_α such that $\dim_u \mu_\alpha = \dim_u L_\alpha^F$, $\mu_\alpha(L_\alpha^F) = 1$, and $\mathscr{P}(\mu_\alpha) = \alpha$. By Birkhoff's ergodic theorem and the ergodicity of μ_α we conclude that there exist the limits

$$\lim_{t\to\infty} \frac{\int_0^t A(\varphi_s x)\, ds}{\int_0^t u(\varphi_s x)\, ds} = \beta(\alpha) = \frac{\int_X A\, d\mu_\alpha}{\int_X u\, d\mu_\alpha},$$

$$\lim_{t\to\infty} \frac{\int_0^t B(\varphi_s x)\, ds}{\int_0^t u(\varphi_s x)\, ds} = \gamma(\alpha) = \frac{\int_X B\, d\mu_\alpha}{\int_X u\, d\mu_\alpha}$$

μ_α-almost everywhere. Therefore $\mu_\alpha(K_{\beta(\alpha),\gamma(\alpha)}) = 1$ and

$$\dim_u K_{\beta(\alpha),\gamma(\alpha)} \geqslant \dim_u \mu_\alpha = \dim_u L_\alpha^F. \tag{16}$$

Furthermore, since $\mathscr{P}(\mu_\alpha) = \alpha$,

$$\left(\frac{\int_X g_1\, d\mu_\alpha}{\int_X h_1\, d\mu_\alpha},\ldots, \frac{\int_X g_d\, d\mu_\alpha}{\int_X h_d\, d\mu_\alpha}\right) = \alpha.$$

On the other hand, again by Birkhoff's ergodic theorem,

$$\left(\frac{\int_X g_1\,d\mu_\alpha}{\int_X h_1\,d\mu_\alpha},\ldots,\frac{\int_X g_d\,d\mu_\alpha}{\int_X h_d\,d\mu_\alpha}\right)=\lim_{t\to\infty}\left(\frac{\int_0^t g_1(\varphi_s x)\,ds}{\int_0^t h_1(\varphi_s x)\,ds},\ldots,\frac{\int_0^t g_d(\varphi_s x)\,ds}{\int_0^t h_d(\varphi_s x)\,ds}\right)$$

$$=\lim_{t\to\infty} F\left(\frac{\int_0^t A(\varphi_s x)\,ds}{\int_0^t u(\varphi_s x)\,ds},\frac{\int_0^t B(\varphi_s x)\,ds}{\int_0^t u(\varphi_s x)\,ds}\right)$$

$$=F\left(\lim_{t\to\infty}\frac{\int_0^t A(\varphi_s x)\,ds}{\int_0^t u(\varphi_s x)\,ds},\lim_{t\to\infty}\frac{\int_0^t B(\varphi_s x)\,ds}{\int_0^t u(\varphi_s x)\,ds}\right)$$

$$=F(\beta(\alpha),\gamma(\alpha)).$$

We have $F(\beta(\alpha),\gamma(\alpha))=\alpha$. Using (15) and (16) we conclude that there exists $(\beta,\gamma)\in F^{-1}(\alpha)$ such that $\mathscr{G}_u(\alpha)=\mathscr{H}_u(\beta,\gamma)$. Therefore,

$$\mathscr{G}_u(\alpha)=\max\{\mathscr{H}_u(\beta,\gamma)\colon(\beta,\gamma)\in F^{-1}(\alpha)\}.$$

Theorem 17 in ref. 6 is a particular case of this example (in the discrete-time case) with $F(X,Y)=X*Y^{-1}$. In this situation $L^F_\alpha=K_\alpha(A,B)$.

5. HYPERBOLIC FLOWS

5.1. Basic Notions

Consider a C^1 flow $\Phi=\{\varphi_t\}_{t\in\mathbb{R}}$ on a manifold M and a compact Φ-invariant hyperbolic set $\Lambda\subset M$ (see Section 1.2 for the definition). The set Λ is called *locally maximal* if it has an open neighborhood U such that $\Lambda=\bigcap_{t\in\mathbb{R}}\varphi_t(U)$. The flow $\Phi|\Lambda$ is called *topologically mixing* if for any two nonempty open sets U and V intersecting Λ, there exists $t\in\mathbb{R}$ such that $\varphi_s(U)\cap V\cap\Lambda\neq\varnothing$ for all $s>t$.

In the case of a hyperbolic set, Proposition 5 can be strengthen in the following manner.

Proposition 9. For a C^1 flow $\Phi=\{\varphi_t\}_{t\in\mathbb{R}}$ on a compact manifold with a locally maximal hyperbolic set Λ such that $\Phi|\Lambda$ is topologically mixing we have:

1. the function $\mu\mapsto h_\mu(\Phi)$ is upper semi-continuous on $\mathscr{M}_\Phi(\Lambda)$;
2. each Hölder continuous function $a\colon\Lambda\to\mathbb{R}$ has a unique equilibrium measure;

3. given Hölder continuous functions $a, b\colon \Lambda \to \mathbb{R}$ the function $\mathbb{R} \ni t \mapsto P_\Phi(a+tb)$ is analytic, and for each $t \in \mathbb{R}$ we have

$$\frac{d^2}{dt^2} P_\Phi(a+tb) \geqslant 0, \tag{17}$$

with equality if and only if b is Φ-cohomologous to a constant.

As a consequence of Proposition 9, the variational principle in Theorem 6 applies in particular to a topologically mixing flow on a locally maximal hyperbolic set.

5.2. Regularity of the Spectrum

The study of the regularity of the multifractal spectrum $\mathcal{F}_u$ is based on statement 2 of Theorem 6 which states that the spectrum is equal to the minimum of a certain function defined implicitly in terms of the topological pressure. Since $t \mapsto P_\Phi(a+tb)$ is analytic when $a, b\colon \Lambda \to \mathbb{R}$ are Hölder continuous functions, we can then use the Implicit function theorem to establish the analyticity of the spectrum.

Theorem 10. Let Φ be a C^1 flow with a compact locally maximal hyperbolic set Λ such that $\Phi \,|\, \Lambda$ is topologically mixing. If the functions $a_i, b_i\colon \Lambda \to \mathbb{R}$ for $i = 1,\ldots, d$ and $u\colon \Lambda \to \mathbb{R}$ are Hölder continuous, then $\mathcal{F}_u$ is analytic on int $\mathcal{P}(\mathcal{M}_\Phi(X))$.

Proof. By Proposition 9, the function $\mu \mapsto h_\mu(\Phi)$ is upper semi-continuous on $\mathcal{M}_\Phi(\Lambda)$. Theorem 6 shows that $\mathcal{F}_u(\alpha) = \min\{T_u(\alpha, q) \colon q \in \mathbb{R}^d\}$, where $T_u(\alpha, q)$ is the unique number satisfying (11). Hence

$$\begin{aligned} 0 &= \partial_q P_\Phi(\langle q, A-\alpha * B\rangle - T_u(\alpha, q)\, u) \\ &= \partial_q P_\Phi(\langle q, A-\alpha * B\rangle - pu)|_{p=T_u(\alpha, q)} \\ &\quad + \partial_p P_\Phi(\langle q, A-\alpha * B\rangle - pu)|_{p=T_u(\alpha, q)}\, \partial_q T_u(\alpha, q). \end{aligned}$$

Consider now $q(\alpha) \in \mathbb{R}^d$ such that $\mathcal{F}_u(\alpha) = T_u(\alpha, q(\alpha))$. Since T_u is of class C^1 (see the discussion after Theorem 6), we have $\partial_q T_u(\alpha, q(\alpha)) = 0$ and thus,

$$\partial_q P_\Phi(\langle q, A-\alpha * B\rangle - pu) = 0,$$

with $q = q(\alpha)$ and $p = T_u(\alpha, q(\alpha))$. Hence, $(\alpha, q, p) = (\alpha, q(\alpha), \mathscr{F}_u(\alpha))$ is a solution of the system

$$\begin{cases} P_\Phi(\langle q, A - \alpha * B\rangle - pu) = 0 \\ \partial_q P_\Phi(\langle q, A - \alpha * B\rangle - pu) = 0. \end{cases} \tag{18}$$

By Proposition 9 the function $t \mapsto P_\Phi(a + tb)$ is analytic. We want to show that

$$\det\left(\frac{\partial[P_\Phi(\langle q, A - \alpha * B\rangle - pu), \partial_q P_\Phi(\langle q, A - \alpha * B\rangle - pu)]}{\partial(q, p)}\right) \neq 0 \tag{19}$$

for $(\alpha, q, p) = (\alpha, q(\alpha), \mathscr{F}_u(\alpha))$. The first line of the matrix in (19) is

$$\left(\partial_q(P_\Phi(\langle q, A - \alpha * B\rangle - pu)), -\int_\Lambda u\, d\mu_\alpha\right),$$

where $\mu_\alpha \in \mathscr{M}_\Phi(\Lambda)$ is the equilibrium measure of $\langle q(\alpha), A - \alpha * B\rangle - \mathscr{F}_u(\alpha)\, u$. Considering the d last equations of the system (18), all the values of the first line vanish at $(\alpha, q(\alpha), \mathscr{F}_u(\alpha))$ except for the last one which is negative.

Therefore, the determinant in (19) will be nonzero provided that

$$\det[\partial_q^2 P_\Phi(\langle q, A - \alpha * B\rangle - pu)] \neq 0 \tag{20}$$

with $(\alpha, q, p) = (\alpha, q(\alpha), \mathscr{F}_u(\alpha))$. We now establish a result analogous to (17).

Lemma 1. Let Φ be a C^1 flow with a compact locally maximal hyperbolic set Λ such that $\Phi | \Lambda$ is topologically mixing. If the functions $a_i, b_i \colon \Lambda \to \mathbb{R}$ for $i = 1, \ldots, d$ and $u \colon \Lambda \to \mathbb{R}$ are Hölder continuous, then the matrix

$$\partial_q^2 P_\Phi(\langle q, A - \alpha * B\rangle - pu) \tag{21}$$

is positive definite for every $q \in \mathbb{R}^d$, $p \in \mathbb{R}$, and $\alpha \in \operatorname{int} \mathscr{P}(\mathscr{M}_\Phi(\Lambda))$.

Proof of Lemma 1. If the determinant of the matrix (21) is zero, then there exists $v \in \mathbb{R}^d \setminus \{0\}$ such that

$$v^t\, \partial_q^2 P_\Phi(\langle q, A - \alpha * B\rangle - pu)\, v = 0,$$

where v^t denotes the transpose of v. Then

$$\partial_t^2 P_\Phi(\langle q - tv, A - \alpha * B\rangle - pu)|_{t=0} = 0$$

and by statement 3 of Proposition 9, $\langle v, A - \alpha * B\rangle$ is Φ-cohomologous to a constant c. In particular,

$$\int_\Lambda \langle v, A - \alpha * B\rangle \, d\mu = \left\langle \int_\Lambda A \, d\mu - \alpha * \int_\Lambda B \, d\mu \right\rangle = c\mu(\Lambda)$$

for every $\mu \in \mathcal{M}_\Phi(\Lambda)$. Since $\alpha \in \mathcal{P}(\mathcal{M}_\Phi(\Lambda))$, this implies that $c = 0$. We conclude that $\langle v, A - \alpha * B\rangle$ is Φ-cohomologous to 0 and thus,

$$P_\Phi(0) = P_\Phi(t\langle v, A - \alpha * B\rangle) \qquad \text{for every} \quad t \in \mathbb{R}.$$

Since $\alpha \in \operatorname{int} \mathcal{P}(\mathcal{M}_\Phi(\Lambda))$, there exists $s \neq 0$ such that $sv + \alpha \in \mathcal{P}(\mathcal{M}_\Phi(\Lambda))$ and there exists a measure $\mu_s \in \mathcal{M}_\Phi(\Lambda)$ satisfying

$$\int_\Lambda A \, d\mu_s = \int_\Lambda (sv + \alpha) * B \, d\mu_s.$$

For every $t \in \mathbb{R}$, we obtain

$$\begin{aligned} P_\Phi(0) &= P_\Phi(t\langle sv, A - \alpha * B\rangle) \\ &\geqslant h_{\mu_s}(\Phi) + t \left\langle sv, (sv + \alpha - \alpha) * \int_\Lambda B \, d\mu_s \right\rangle \\ &\geqslant ts^2 \, |v|^2 \inf_{i \in \{1, \dots, d\}} \inf b_i, \end{aligned}$$

which is impossible (let $t \to \infty$). We conclude that the matrix (21) has nonzero determinant. We now show that it is positive definite. By the continuity of

$$v \mapsto v^t \, \partial_q^2 P_\Phi(\langle q, A - \alpha * B\rangle - pu) \, v,$$

if we had vectors $v = (v_1, \dots, v_d)$ and $w = (w_1, \dots, w_d) \in \mathbb{R}^d \setminus \{0\}$ satisfying

$$v^t \, \partial_q^2 P_\Phi(\langle q, A - \alpha * B\rangle - pu) \, v < 0 \; \text{ and } \; w^t \, \partial_q^2 P_\Phi(\langle q, A - \alpha * B\rangle - pu) \, w > 0,$$

we could find $t_1, \dots, t_d \in (0, 1)$ such that

$$x = (t_1 v_1 + (1 - t_1) \, w_1, \dots, t_d v_d + (1 - t_d) \, w_d) \neq 0$$

and

$$x^t\, \partial_q^2 P_\Phi(\langle q, A - \alpha * B\rangle - pu)\, x = 0.$$

But as shown above this is impossible. Therefore, the matrix (21) is either positive definite or negative definite.

Denote by e_1 the first element of the canonical base of $\mathbb{R}^d$. By statement 3 of Proposition 9,

$$e_1^t\, \partial_q^2 P_\Phi(\langle q, A - \alpha * B\rangle - pu)\, e_1 = \frac{\partial^2}{\partial q_1^2} P_\Phi(\langle q, A - \alpha * B\rangle - pu) \geqslant 0$$

and we conclude that the matrix in (21) is positive definite. ∎

Lemma 20 shows that (20) holds. Consequently the system (18) defines q and p as functions of α in a neighborhood of $(\alpha, q(\alpha), \mathscr{F}_u(\alpha))$. In particular, we conclude that the spectrum $\mathscr{F}_u$ is analytic on int $\mathscr{P}(\mathscr{M}_\Phi(\Lambda))$. ∎

5.3. Variational Properties

We now study how the spectrum $\mathscr{F}_u$ varies when the vectors A, B and the function u are perturbed. The results obtained here are consequences of the Conditional variational principle and of Theorem 10. We use a similar approach to that of Barreira in ref. 2.

Consider a C^1 flow $\Phi = \{\varphi_t\}_{t \in \mathbb{R}}$ on a manifold M and Λ a compact locally maximal hyperbolic set for Φ such that $\Phi \mid \Lambda$ is topologically mixing.

Let $C^\varepsilon(\Lambda)$ be the space of the Hölder continuous functions $a\colon \Lambda \to \mathbb{R}$ with Hölder exponent ε. We define the norm of a function $a \in C^\varepsilon(\Lambda)$ by

$$\|a\|_\varepsilon = \sup\{|a(x)|\colon x \in \Lambda\} + \sup\left\{\frac{|a(x) - a(y)|}{d(x, y)^\varepsilon}\colon x, y \in \Lambda \text{ and } x \neq y\right\}.$$

Consider a family of functions $a\colon \Lambda \times (-\delta, \delta) \to \mathbb{R}$ in $C^\varepsilon(\Lambda)$, $\delta > 0$. We write $a(\eta)$ when we refer to the function in this family corresponding to a specific value $\eta \in (\delta, \delta)$. We say that a is a C^k-*family* if the map $(-\delta, \delta) \ni \eta \mapsto a(\eta) \in C^\varepsilon(\Lambda)$ is of class C^k.

Consider now two vectors

$$A\colon \Lambda \times (-\delta, \delta) \to \mathbb{R}^d \qquad \text{and} \qquad B\colon \Lambda \times (-\delta, \delta) \to \mathbb{R}^d$$

in $(C^\varepsilon(\Lambda))^d$ whose coordinates are C^k-families defined for $\eta \in (-\delta, \delta)$. We write $A(\eta)$ when we want to refer to the vector corresponding to a specific value $\eta \in (\delta, \delta)$. Consider also a C^k-family u defined for $\eta \in (-\delta, \delta)$.

Given $q \in \mathbb{R}^d$, let $T_u(\alpha, q, \eta)$ be the unique real number such that

$$P_\Phi(\langle q, A(\eta) - \alpha * B(\eta)\rangle - T_u(\alpha, q, \eta)\, u(\eta)) = 0. \tag{22}$$

We denote by $\mu_{q,\eta}$ the unique equilibrium measure of

$$\langle q, A(\eta) - \alpha * B(\eta)\rangle - T_u(\alpha, q, \eta)\, u(\eta).$$

Theorem 11. Consider a C^1 flow $\Phi = \{\varphi_t\}_{t\in\mathbb{R}}$, Λ a compact locally maximal hyperbolic set for Φ such that $\Phi \mid \Lambda$ is topologically mixing, A and B two vectors of $(C^\varepsilon(\Lambda))^d$ whose coordinates are C^k-families and u a C^k-family in $C^\varepsilon(\Lambda)$ such that $u(0) > 0$ (for some $k \geqslant 1$). Then the function $(\alpha, q, \eta) \mapsto T_u(\alpha, q, \eta)$ is of class C^k in η and analytic in α and q in a neighborhood of $\eta = 0$, with

$$\left.\frac{\partial T_u}{\partial \eta}\right|_{(\alpha, q, \eta) = (\alpha, q, 0)} = \frac{\langle q, \int_\Lambda \zeta \, d\mu_{q,0}\rangle - T_u(\alpha, q, 0) \int_\Lambda \left.\frac{du}{d\eta}\right|_{\eta=0} d\mu_{q,0}}{\int_\Lambda u(0)\, d\mu_{q,0}} \tag{23}$$

for every $q \in \mathbb{R}^d$, where

$$\zeta = \left.\frac{dA}{d\eta}\right|_{\eta=0} - \alpha * \left.\frac{dB}{d\eta}\right|_{\eta=0}.$$

Proof. Consider the equation

$$P_\Phi(\langle q, A - \alpha * B\rangle - pu) = 0. \tag{24}$$

By the definition of $T_u(\alpha, q, 0)$, the vector

$$(p, \alpha, q, \eta) = (T_u(\alpha, q, 0), \alpha, q, 0)$$

is a solution of (24). Furthermore, the first member of (24) is of class C^k in η and analytic in p, α, and q. Since

$$\left.\frac{\partial}{\partial p} P_\Phi(\langle q, A - \alpha * B\rangle - pu)\right|_{(p, \alpha, q, \eta) = (T_u(\alpha, q, 0), \alpha, q, 0)} = -\int_\Lambda u(0)\, d\mu_{q,0} < 0,$$

we conclude that (24) implicitly defines p as function of α, q, and η in a neighborhood of $(T_u(\alpha, q, 0), \alpha, q, 0)$. We obtain a function $T_u\colon (-\delta', \delta') \times \mathbb{R}^d \to \mathbb{R}$ with the desired regularity, for some $\delta' \in (0, \delta]$.

Consider now the Taylor expansion around $\eta=0$,

$$\begin{aligned}\langle q, A(\eta)-\alpha * B(\eta)\rangle - T_u(\alpha, q, \eta)\, u(\eta) \\ = \langle q, A(0)-\alpha * B(0)\rangle - T_u(\alpha, q, 0)\, u(0) + \left\langle q, \frac{dA}{d\eta}\Big|_{\eta=0} - \alpha * \frac{dB}{d\eta}\Big|_{\eta=0}\right\rangle \eta \\ - \frac{\partial T_u}{\partial \eta}\Big|_{\eta=0} u(0)\, \eta - T_u(\alpha, q, 0) \frac{du}{d\eta}\Big|_{\eta=0} \eta + o(\eta).\end{aligned}$$

By (22), we have

$$\begin{aligned}0 &= \frac{\partial}{\partial \eta} P_\Phi(\langle q, A(\eta)-\alpha * B(\eta)\rangle - T_u(\alpha, q, \eta)\, u(\eta))\Big|_{\eta=0} \\ &= \int_\Lambda \left(\left\langle q, \frac{dA}{d\eta}\Big|_{\eta=0} - \alpha * \frac{dB}{d\eta}\Big|_{\eta=0}\right\rangle - T_u(\alpha, q, 0)\frac{du}{d\eta}\Big|_{\eta=0}\right) d\mu_{q,0} \\ &\quad - \frac{\partial T_u}{\partial \eta}\Big|_{\eta=0} \int_\Lambda u(0)\, d\mu_{q,0}.\end{aligned}$$

This establishes (23). ∎

The following theorem describes how the spectrum $\mathcal{F}_u = \mathcal{F}_u(\alpha, \eta)$ changes when η varies.

Theorem 12. Consider a C^1 flow $\Phi = \{\varphi_t\}_{t\in\mathbb{R}}$, Λ a compact locally maximal hyperbolic set for Φ such that $\Phi \mid \Lambda$ is topologically mixing, A and B two vectors of $(C^\varepsilon(\Lambda))^d$ whose coordinates are C^k-families and u a C^k-family in $C^\varepsilon(\Lambda)$ such that $u(0)>0$ (for some $k \geqslant 1$). Then

1. the spectrum $\mathcal{F}_u$ is of class C^k in η in a neighborhood of $\eta=0$ and analytic in α on int $\mathcal{P}(M_\Phi(\Lambda))$;

2. if for each $i \in \{1,..., d\}$ and $\eta \in (-\delta, \delta)$ the function $b_i(\eta)$ is Φ-cohomologous to $\gamma_i(\eta)\, u(\eta)$, with $\gamma = (\gamma_1,..., \gamma_d)\colon (-\delta, \delta) \to \mathbb{R}^d$ of class C^k on η and independent of $x \in \Lambda$, we have

$$\frac{\partial \mathcal{F}_u}{\partial \eta}\Big|_{(\alpha,\eta)=(\gamma(0)^{-1} * \partial_q Q_u(q,0),\, 0)} = \frac{\langle q, \int_\Lambda \xi\, d\mu_{q,0}\rangle - Q_u(q,0) \int_\Lambda \frac{du}{d\eta}\big|_{\eta=0} d\mu_{q,0}}{\int_\Lambda u(0)\, d\mu_{q,0}}, \qquad (25)$$

for every $q \in \mathbb{R}^d$, where $Q_u(q, \eta)$ is the unique number satisfying

$$P_\Phi(\langle q, A(\eta)\rangle - Q_u(q,\eta)\, u(\eta)) = 0,$$

and where

$$\xi=\frac{dA}{d\eta}\bigg|_{\eta=0}-\gamma(0)^{-1}*\partial_q Q_u(q,0)*\frac{dB}{d\eta}\bigg|_{\eta=0}+\partial_q Q_u(q,0)\,\frac{du}{d\eta}\bigg|_{\eta=0}.$$

Proof. To prove the first statement we consider the system

$$\begin{cases} P_\Phi(\langle q, A-\alpha*B\rangle - pu)=0\\ \partial_q P_\Phi(\langle q, A-\alpha*B\rangle - pu)=0. \end{cases} \tag{26}$$

Using a similar approach to that in the proof of Theorem 10, and applying the Implicit function theorem around $(p,\alpha,q,\eta)=(\mathcal{F}_u(\alpha,0),\alpha,q(\alpha),0)$, we conclude that the spectrum $\mathcal{F}_u$ possesses the desired regularity.

We consider the identity

$$\partial_q Q_u(q,\eta)=\alpha*\gamma(\eta)+\partial_q T_u(\alpha,q,\eta). \tag{27}$$

Taking derivatives with respect to q, we obtain $\partial_q^2 Q_u(q,\eta)=\partial_q^2 T_u(\alpha,q,\eta)$. Hence,

$$\begin{aligned}\det\partial_q^2(\gamma(\eta)^{-1}*\partial_q Q_u(q,\eta)) &= \gamma_1(\eta)^{-1}\cdots\gamma_d(\eta)^{-1}\det\partial_q^2 Q_u(q,\eta)\\ &= \gamma_1(\eta)^{-1}\cdots\gamma_d(\eta)^{-1}\det\partial_q^2 T_u(\alpha,q,\eta).\end{aligned} \tag{28}$$

Let now $\eta\in(-\delta,\delta)$. By (22) we have

$$\begin{aligned} 0 &= \partial_q P_\Phi(\langle q, A(\eta)-\alpha*B(\eta)\rangle - T_u(\alpha,q,\eta)\,u(\eta))\\ &= \partial_q P_\Phi(\langle q, A(\eta)-\alpha*B(\eta)\rangle - pu(\eta))|_{p=T_u(\alpha,q,\eta)}\\ &\quad+\partial_p P_\Phi(\langle q, A(\eta)-\alpha*B(\eta)\rangle - pu(\eta))|_{p=T_u(\alpha,q,\eta)}\,\partial_q T_u(\alpha,q,\eta). \end{aligned}$$

This allow us to conclude that

$$\partial_q T_u(\alpha,q,\eta)=\frac{\partial_q P_\Phi(\langle q, A(\eta)-\alpha*B(\eta)\rangle - pu(\eta))|_{p=T_u(\alpha,q,\eta)}}{\int_\Lambda u(\eta)\,d\mu_{q,\eta}}. \tag{29}$$

The ith component of $\partial_q T_u(\alpha,q,\eta)$ is

$$(\partial_q T_u(\alpha,q,\eta))_i=\frac{\frac{\partial}{\partial q_i}P_\Phi(\langle q, A(\eta)-\alpha*B(\eta)\rangle - pu(\eta))|_{p=T_u(\alpha,q,\eta)}}{\int_\Lambda u(\eta)\,d\mu_{q,\eta}}.$$

Considering $j \in \{1,\ldots,d\}$ and taking derivatives with respect to q_j, we obtain

$$\begin{aligned}
\frac{\partial}{\partial q_j}(\partial_q T_u(\alpha, q, \eta))_i
&= \frac{\frac{\partial^2}{\partial q_j\,\partial q_i} P_\Phi(\langle q, A(\eta)-\alpha * B(\eta)\rangle - pu(\eta))|_{p=T_u(\alpha,q,\eta)}}{\int_\Lambda u(\eta)\,d\mu_{q,\eta}} \\
&+ \frac{\frac{\partial^2}{\partial p\,\partial q_i} P_\Phi(\langle q, A(\eta)-\alpha * B(\eta)\rangle - pu(\eta))|_{p=T_u(\alpha,q,\eta)} \frac{\partial}{\partial q_j} T_u(\alpha, q, \eta)}{\int_\Lambda u(\eta)\,d\mu_{q,\eta}} \\
&- \frac{\frac{\partial}{\partial q_i} P_\Phi(\langle q, A(\eta)-\alpha * B(\eta)\rangle - pu(\eta))|_{p=T_u(\alpha,q,\eta)} \frac{\partial}{\partial q_j} \int_\Lambda u(\eta)\,d\mu_{q,\eta}}{(\int_\Lambda u(\eta)\,d\mu_{q,\eta})^2}. \qquad (30)
\end{aligned}$$

By the discussion at the end of Section 3.1, we have

$$\mathcal{F}_u(\gamma(\eta)^{-1} * \partial_q Q_u(q,\eta), \eta) = Q_u(q,\eta) - \langle q, \partial_q Q_u(q,\eta)\rangle, \qquad (31)$$

where q satisfies $\partial_q T_u(\alpha, q, \eta) = 0$. By (29),

$$\partial_q P_\Phi(\langle q, A(\eta)-\alpha * B(\eta)\rangle - pu(\eta))|_{p=T_u(\alpha,q,\eta)} = 0,$$

and (30) simplifies to give

$$\frac{\frac{\partial^2}{\partial q_j\,\partial q_i} P_\Phi(\langle q, A(\eta)-\alpha * B(\eta)\rangle - pu(\eta))|_{p=T_u(\alpha,q,\eta)}}{\int_\Lambda u(\eta)\,d\mu_{q,\eta}}.$$

Thus $\det \partial_q^2 T_u(\alpha, q, \eta)$ equals

$$\left(\frac{1}{\int_\Lambda u(\eta)\,d\mu_{q,\eta}}\right)^d \det \partial_q^2 P_\Phi(\langle q, A(\eta)-\alpha * B(\eta)\rangle - pu(\eta))|_{p=T_u(\alpha,q,\eta)}.$$

By Lemma 1, we obtain $\det \partial_q^2 T_u(\alpha, q, \eta) \neq 0$. Using (28), we conclude that

$$\det \partial_q^2(\gamma(\eta)^{-1} * \partial_q Q_u(q,\eta)) \neq 0.$$

Thus $\gamma^{-1} * \partial_q Q_u$ is locally invertible, and hence, given $q \in \mathbb{R}^d$ and $\alpha = \gamma(\eta)^{-1} * \partial_q Q_u(q,\eta)$ (see the end of Section 3.1), there exists a family of functions G: $\mathbb{R}^d \times (-\delta, \delta) \to \mathbb{R}^d$, depending on η, such that

$$\gamma(\eta)^{-1} * \partial_q Q_u(G(\alpha,\eta),\eta) = \alpha \qquad \text{and} \qquad G(\gamma(\eta)^{-1} * \partial_q Q_u(q,\eta),\eta) = q.$$

Since Q_u is of class C^k in η, the same happens with G. We have

$$\mathscr{F}_u(\alpha, \eta) = Q_u(G(\alpha, \eta), \eta) - \langle G(\alpha, \eta), \alpha * \gamma(\eta)\rangle$$

and taking derivatives with respect to η we obtain

$$\begin{aligned}\frac{\partial \mathscr{F}_u}{\partial \eta}\bigg|_{(\alpha,\eta)=(\alpha,0)} &= \frac{\partial Q_u}{\partial \eta}\bigg|_{(q,\eta)=(G(\alpha,0),0)} + \left\langle \partial_q Q_u(G(\alpha,0),0), \frac{\partial G}{\partial \eta}\bigg|_{(\alpha,\eta)=(\alpha,0)} \right\rangle \\ &\quad - \left\langle \frac{\partial G}{\partial \eta}\bigg|_{(\alpha,\eta)=(\alpha,0)}, \alpha * \gamma(0) \right\rangle - \left\langle G(\alpha,0), \alpha * \frac{d\gamma}{d\eta}\bigg|_{\eta=0} \right\rangle.\end{aligned}$$

Since q satisfies $\partial_q T_u(\alpha, q, \eta) = 0$, it follows from (27) that

$$\frac{\partial \mathscr{F}_u}{\partial \eta}\bigg|_{(\alpha,\eta)=(\alpha,0)} = \frac{\partial Q_u}{\partial \eta}\bigg|_{(q,\eta)=(G(\alpha,0),0)} - \left\langle G(\alpha,0), \alpha * \frac{d\gamma}{d\eta}\bigg|_{\eta=0} \right\rangle.$$

Furthermore

$$\frac{\partial Q_u}{\partial \eta}\bigg|_{(q,\eta)=(q,0)} = \left\langle q, \alpha * \frac{d\gamma}{d\eta}\bigg|_{\eta=0} \right\rangle + \frac{\partial T_u}{\partial \eta}\bigg|_{(q,\eta)=(q,0)}$$

for every $q \in \mathbb{R}^d$, and thus

$$\begin{aligned}\frac{\partial \mathscr{F}_u}{\partial \eta}\bigg|_{(\alpha,\eta)=(\alpha,0)} &= \frac{\partial Q_u}{\partial \eta}\bigg|_{(q,\eta)=(G(\alpha,0),0)} - \left\langle G(\alpha,0), \alpha * \frac{d\gamma}{d\eta}\bigg|_{\eta=0} \right\rangle \\ &= \frac{\partial T_u}{\partial \eta}\bigg|_{(q,\eta)=(G(\alpha,0),0)}.\end{aligned}$$

It now follows from (23) that

$$\frac{\partial \mathscr{F}_u}{\partial \eta}\bigg|_{(\alpha,\eta)=(\alpha,0)} = \frac{\langle G(\alpha,0), \int_\Lambda \zeta \, d\mu_{q,0}\rangle - T_u(\alpha, G(\alpha,0), 0) \int_\Lambda \frac{du}{d\eta}\big|_{\eta=0} \, d\mu_{q,0}}{\int_\Lambda u(0)\, d\mu_{q,0}}.$$

Taking $\alpha = \gamma(0)^{-1} * \partial_q Q_u(q, 0)$ and since $T_u(\alpha, q, 0) = Q_u(q, 0) - \langle q, \alpha * \gamma(0)\rangle$, we obtain (25). This completes the proof of the theorem. ∎

We note that statement 2 of Theorem 12 could be obtained from the Implicit function theorem applied to the system (26). Then

$$\left.\frac{\partial \mathscr{F}_u}{\partial \eta}\right|_{(\alpha,\eta)=(\alpha,0)} = -\left.\frac{\det\left(\frac{\partial[P_\Phi(\langle q, A-\alpha * B\rangle - pu), \partial_q P_\Phi(\langle q, A-\alpha * B\rangle - pu)]}{\partial(q,\eta)}\right)}{\det\left(\frac{\partial[P_\Phi(\langle q, A-\alpha * B\rangle - pu), \partial_q P_\Phi(\langle q, A-\alpha * B\rangle - pu)]}{\partial(q,p)}\right)}\right|_{(p,\alpha,q,\eta)=(\mathscr{F}_u(\alpha,0),\alpha,q(\alpha),0)}.$$

When $(p, \alpha, q, \eta) = (\mathscr{F}_u(\alpha, 0), \alpha, q(\alpha), 0)$, we have

$$\det\left(\frac{\partial[P_\Phi(\langle q, A-\alpha * B\rangle - pu), \partial_q P_\Phi(\langle q, A-\alpha * B\rangle - pu)]}{\partial(q,\eta)}\right)$$
$$= \partial_\eta P_\Phi(\langle q, A-\alpha * B\rangle - pu)\cdot \det(\partial_q^2 P_\Phi(\langle q, A-\alpha * B\rangle - pu))$$
$$= \int_\Lambda \left(\left\langle q, \frac{dA}{d\eta} - \alpha * \frac{dB}{d\eta}\right\rangle - p\frac{du}{d\eta}\right) d\mu_{\alpha,0}\cdot \det(\partial_q^2 P_\Phi(\langle q, A-\alpha * B\rangle - pu))$$

and

$$\det\left(\frac{\partial[P_\Phi(\langle q, A-\alpha * B\rangle - pu), \partial_q P_\Phi(\langle q, A-\alpha * B\rangle - pu)]}{\partial(q,p)}\right)$$
$$= -\int_\Lambda u\, d\mu_{\alpha,0}\cdot \det(\partial_q^2 P_\Phi(\langle q, A-\alpha * B\rangle - pu)).$$

We obtain

$$\left.\frac{\partial \mathscr{F}_u}{\partial \eta}\right|_{(\alpha,\eta)=(\alpha,0)} = \frac{\langle q, \int_\Lambda \zeta\, d\mu_{\alpha,0}\rangle - \mathscr{F}_u(\alpha,0)\int_\Lambda \frac{du}{d\eta}\big|_{\eta=0}\, d\mu_{\alpha,0}}{\int_\Lambda u(0)\, d\mu_{\alpha,0}},$$

where $\mu_{\alpha,0}$ is the unique equilibrium measure of

$$\langle q(\alpha), A(0) - \alpha * B(0)\rangle - \mathscr{F}_u(\alpha,0)\, u(0).$$

If there is a cohomology relation between B and u for each $\eta \in (-\delta, \delta)$ (as considered in statement 2 of Theorem 12) then (31) holds. This allows us to conclude that (25) also holds.

For the constant family $u = 1$, Theorem 12 describes how the *entropy spectrum* $\mathscr{E}(\alpha) = \mathscr{F}_1(\alpha) = h(\Phi \mid K_\alpha)$ varies with η.

Corollary 13. Consider a C^1 flow $\Phi = \{\varphi_t\}_{t\in\mathbb{R}}$, Λ a compact locally maximal hyperbolic set for Φ such that $\Phi \mid \Lambda$ is topologically mixing, and

A and B two vectors of $(C^{\varepsilon}(\Lambda))^d$ whose coordinates are C^k-families of functions.

1. the entropy spectrum $\mathscr{E}$ is of class C^k in η in a neighborhood of $\eta = 0$ and analytic in α on $\operatorname{int} \mathscr{P}(M_\Phi(\Lambda))$;

2. if, for each $i \in \{1,..., d\}$ and $\eta \in (-\delta, \delta)$, the function $b_i(\eta)$ is Φ-cohomologous to $\gamma_i(\eta)$, with $\gamma = (\gamma_1,..., \gamma_d): (-\delta, \delta) \to \mathbb{R}^d$ of class C^k on η and independent of $x \in \Lambda$, we have

$$\left.\frac{\partial \mathscr{E}}{\partial \eta}\right|_{(\alpha, \eta) = (\gamma(0)^{-1} * \partial_q Q_1(q, 0), 0)}$$

$$= \left\langle q, \int_\Lambda \left.\frac{dA}{d\eta}\right|_{\eta=0} \right\rangle - \left\langle q, \gamma(0)^{-1} * \partial_q Q_1(q, 0) * \int_\Lambda \left.\frac{dB}{d\eta}\right|_{\eta=0} d\mu_{q,0} \right\rangle,$$

for every $q \in \mathbb{R}^d$, where $Q_1(q, \eta) = P_\Phi(\langle q, A(\eta)\rangle)$.

6. DIMENSION SPECTRA

The purpose of this section is to discuss the properties of dimension spectra. These spectra are obtained by computing the Hausdorff dimension of level sets of Birkhoff averages (both for positive and negative time).

Let $\Phi = \{\varphi_t\}_{t \in \mathbb{R}}$ be a C^1 flow on a manifold M and $\Lambda \subset M$ a compact Φ-invariant locally maximal hyperbolic set. For each $x \in \Lambda$ there exist *stable* and *unstable local manifolds* $V^s(x)$ and $V^u(x)$ containing x such that:

1. $T_x V^s(x) = E^s(x)$ and $T_x V^u(x) = E^u(x)$;
2. $\varphi_t(V^s(x)) \subset V^s(\varphi_t x)$ and $\varphi_{-t}(V^u(x)) \subset V^u(\varphi_{-t} x)$ for every $t > 0$;
3. there exist $\kappa > 0$ and $\mu > 0$ such that for each $t \geqslant 0$,

$$\begin{aligned} d(\varphi_t y, \varphi_t x) &\leqslant \kappa e^{-\mu t} d(y, x) \quad \text{whenever} \quad y \in V^s(x), \\ d(\varphi_{-t} y, \varphi_{-t} x) &\leqslant \kappa e^{-\mu t} d(y, x) \quad \text{whenever} \quad y \in V^u(x). \end{aligned} \tag{32}$$

The flow Φ is said to be *conformal* on Λ if the maps

$$d_x\varphi_t \mid E^s(x): E^s(x) \to E^s(\varphi_t x) \qquad \text{and} \qquad d_x\varphi_t \mid E^u(x): E^u(x) \to E^u(\varphi_t x)$$

are multiples of isometries for each $x \in \Lambda$ and $t \in \mathbb{R}$.

Given functions $(A, B) \in C(\Lambda)^d \times C(\Lambda)^d$ with $B > 0$, we consider the level sets K_α in (8). We also consider Birkhoff averages for negative time. Namely, given $e \in \mathbb{N}$ and vectors $(F, G) \in C(\Lambda)^e \times C(\Lambda)^e$,

$$F = (f_1,..., f_e) \qquad \text{and} \qquad G = (g_1,..., g_e),$$

with $G>0$, for each $\beta=(\beta_1,\dots,\beta_e)\in\mathbb{R}^e$ we let

$$M_\beta=\bigcap_{i=1}^{e}\left\{x\in\Lambda:\ \lim_{t\to-\infty}\frac{\int_0^t f_i(\varphi_s x)\,ds}{\int_0^t g_i(\varphi_s x)\,ds}=\beta_i\right\}.$$

We define the *dimension spectrum* $\mathscr{D}$: $\mathbb{R}^d\times\mathbb{R}^e\to\mathbb{R}$ (with respect to Φ) by

$$\mathscr{D}(\alpha,\beta)=\dim_H(K_\alpha\cap M_\beta), \tag{33}$$

where $\dim_H Z$ denotes the Hausdorff dimension of the set Z.

Theorem 14. Let Φ be a $C^{1+\alpha}$ flow with a compact locally maximal hyperbolic set Λ on which Φ is conformal, and functions $(A,B)\in C(\Lambda)^d\times C(\Lambda)^d$ with $B>0$, and $(F,G)\in C(\Lambda)^e\times C(\Lambda)^e$ with $G>0$. Then the following properties hold:

1. for each $\alpha\in\mathbb{R}^d$, $\beta\in\mathbb{R}^e$, $x\in K_\alpha$, and $y\in M_\beta$ we have

$$\dim_H K_\alpha=\dim_H(K_\alpha\cap V^u(x))+2=\dim_v K_\alpha+2,$$
$$\dim_H M_\beta=\dim_H(M_\beta\cap V^s(y))+2=\dim_w M_\beta+2,$$

where

$$v(x)=\frac{\partial}{\partial t}\log\|d_x\varphi_t\,|\,E^u(x)\|\Big|_{t=0},\qquad w(x)=-\frac{\partial}{\partial t}\log\|d_x\varphi_t\,|\,E^s(x)\|\Big|_{t=0}; \tag{34}$$

2. for each $\alpha\in\mathbb{R}^d$ and $\beta\in\mathbb{R}^e$ we have

$$\mathscr{D}(\alpha,\beta)=\dim_H K_\alpha+\dim_H M_\beta-3=\dim_v K_\alpha+\dim_w M_\beta+1. \tag{35}$$

Proof. Let a: $\Lambda\to\mathbb{R}$ be a continuous function. It follows from (32) and from the uniform continuity of a on Λ that for each $x\in\Lambda$ and $\delta>0$ there exists $t_0>0$ such that if $y\in V^s(x)$ and $t>t_0$ then

$$\left|\frac{1}{t}\int_{t_0}^{t}a(\varphi_s y)\,ds-\frac{1}{t}\int_{t_0}^{t}a(\varphi_s x)\,ds\right|<\delta.$$

This implies that $V^s(x)\subset K_\alpha$ for every $x\in K_\alpha$. Furthermore, the set K_α is Φ-invariant and thus $\bigcup_{t\in\mathbb{R}}\varphi_t(V^s(x))\subset K_\alpha$ whenever $x\in K_\alpha$.

Since Φ is conformal on Λ, it follows from work of Hasselblatt in ref. 11 that the weak stable distribution $x\mapsto E^s(x)\oplus E^0(x)$ and the weak unstable distribution $x\mapsto E^u(x)\oplus E^0(x)$ are Lipschitz (as observed by Pesin and Sadovskaya in ref. 15). This ensures that in a sufficiently small open neighborhood of $x\in K_\alpha$ the set K_α is taken by a Lipschitz map with

Lipschitz inverse onto $\bigcup_{t\in I}\varphi_t(V^s(x))\times V^u(x)$, where I is some open interval containing zero. Therefore,

$$\dim_H K_\alpha = \dim_H(K_\alpha \cap V^u(x)) + 2$$

(the detailed proof uses the fact that the Hausdorff dimension and the upper box dimension of $K_\alpha \cap V^u(x)$ coincide, also in view of the conformality assumption; we refer to refs. 1 and 15 for the detailed argument).

For the second identity, we first note that

$$\int_0^t v(\varphi_s x)\, ds = \log \|d_x\varphi_t \mid E^u(x)\|.$$

Since the distribution $x \mapsto E^u(x)$ is Hölder continuous on Λ (in fact it is Lipschitz in our context) and Φ is of class $C^{1+\alpha}$, the function v is Hölder continuous and for each $\varepsilon > 0$ there exist constants $c_1, c_2 > 0$ such that

$$c_1 \exp(-\alpha v(x, t, \varepsilon)) \leqslant [\operatorname{diam}(B_\varepsilon(x, t) \cap V^u(x))]^\alpha \leqslant c_2 \exp(-\alpha v(x, t, \varepsilon)).$$

This ensures that $\dim_H(Z \cap V^u(x)) = \dim_v Z$ for every set $Z \subset \Lambda$. The second identity is obtained by setting $Z = K_\alpha$. The corresponding statement concerning M_β is established in an analogous manner.

To prove the second statement we first note that given $x \in K_\alpha \cap M_\beta$ and a sufficiently small open neighborhood U of x we have $K_\alpha \cap U = \bigcup_{y\in K_\alpha\cap U} V^s(y)$ and $M_\beta \cap U = \bigcup_{y\in M_\beta\cap U} V^u(y)$. Therefore,

$$K_\alpha \cap M_\beta \cap U = \bigcup_{y\in K_\alpha\cap U} V^s(y) \cap \bigcup_{y\in M_\beta\cap U} V^u(y).$$

Since the weak stable and unstable distributions are Lipschitz, this identity implies that for a sufficiently small codimension-one disc $D \subset U$ transverse to the flow, the set $K_\alpha \cap M_\beta \cap D$ is taken by a Lipschitz map with Lipschitz inverse onto the product $(K_\alpha \cap V^u(x)) \times (M_\beta \cap V^s(x))$. Therefore, using the fact that the Hausdorff dimension and the upper box dimension of $K_\alpha \cap V^u(x)$ coincide (and analogously for $M_\beta \cap V^s(x)$),

$$\dim_H(K_\alpha \cap M_\beta \cap D) = \dim_H(K_\alpha \cap V^u(x)) + \dim_H(M_\beta \cap V^s(x)).$$

Since $K_\alpha \cap M_\beta$ is Φ-invariant, we obtain

$$\begin{aligned}\dim_H(K_\alpha \cap M_\beta \cap U) &= \dim_H(K_\alpha \cap M_\beta \cap D) + 1\\ &= \dim_H(K_\alpha \cap V^u(x)) + \dim_H(M_\beta \cap V^s(x)) + 1\\ &= \dim_H K_\alpha + \dim_H M_\beta - 3.\end{aligned}$$

This completes the proof. ∎

We note that a version of the formula in (35) was obtained by Wolf in ref. 21, although in a very different context (namely for hyperbolic polynomial automorphisms of $\mathbb{C}^2$).

Consider now the functions $\mathscr{P}\colon \mathscr{M}_\Phi(\Lambda) \to \mathbb{R}^d$ in (9) and $\mathscr{Q}\colon \mathscr{M}_\Phi(\Lambda) \to \mathbb{R}^e$ defined by

$$\mathscr{Q}(\mu) = \left(\frac{\int_\Lambda f_1\, d\mu}{\int_\Lambda g_1\, d\mu}, \ldots, \frac{\int_\Lambda f_e\, d\mu}{\int_\Lambda g_e\, d\mu} \right).$$

The following is an immediate consequence of Theorems 6, 10, and 14.

Corollary 15. Let Φ be a $C^{1+\alpha}$ flow with a compact locally maximal hyperbolic set Λ on which Φ is conformal, and functions $(A, B) \in C^\varepsilon(\Lambda)^d \times C^\varepsilon(\Lambda)^d$ with $B > 0$, and $(F, G) \in C^\varepsilon(\Lambda)^e \times C^\varepsilon(\Lambda)^e$ with $G > 0$. Then the following properties hold:

1. if $\alpha \in \operatorname{int} \mathscr{P}(\mathscr{M}_\Phi(\Lambda))$ and $\beta \in \operatorname{int} \mathscr{Q}(\mathscr{M}_\Phi(\Lambda))$ then $K_\alpha \cap M_\beta \neq \varnothing$ and

$$\mathscr{D}(\alpha, \beta) = \max\left\{ \frac{h_\mu(\Phi)}{\int_\Lambda v\, d\mu} : \mu \in \mathscr{M}_\Phi(\Lambda) \text{ and } \mathscr{P}(\mu) = \alpha \right\}$$
$$+ \max\left\{ \frac{h_\mu(\Phi)}{\int_\Lambda w\, d\mu} : \mu \in \mathscr{M}_\Phi(\Lambda) \text{ and } \mathscr{Q}(\mu) = \beta \right\} + 1,$$

where v and w are as in (34);

2. the spectrum $\mathscr{D}$ is analytic on $\operatorname{int} \mathscr{P}(\mathscr{M}_\Phi(\Lambda)) \times \operatorname{int} \mathscr{Q}(\mathscr{M}_\Phi(\Lambda))$.

Proof. It remains to observe that since the distributions $x \mapsto E^s(x)$ and $x \mapsto E^u(x)$ are Hölder continuous on Λ, the functions v and w are also Hölder continuous. ∎

We now consider the particular case when the Birkhoff averages are obtained from the Lyapunov exponents. We continue to assume that the flow Φ is conformal on Λ. Let also μ be a Φ-invariant probability measure on Λ. By Birkhoff's ergodic theorem, for μ-almost every $x \in \Lambda$ there exist the limits

$$\lambda_s(x) = \lim_{t \to +\infty} \frac{1}{t} \log \|d_x\varphi_t \mid E^s(x)\| \quad \text{and} \quad \lambda_u(x) = \lim_{t \to +\infty} \frac{1}{t} \log \|d_x\varphi_t \mid E^u(x)\|.$$

These are the two values of the Lyapunov exponent at x. As observed by Pesin and Sadovskaya in ref. 15,

$$\lambda_s(x) = \lim_{t \to +\infty} -\frac{1}{t} \int_0^t w(\varphi_\tau x)\, d\tau \qquad \text{and} \qquad \lambda_u(x) = \lim_{t \to +\infty} \frac{1}{t} \int_0^t v(\varphi_\tau x)\, d\tau.$$

We set $d=e=2$, and consider the pairs of functions $A=F=(-w,1)$ and $B=G=(v,1)$. We also consider the associated dimension spectrum $\mathscr{D}$ in (33). Set $\mathscr{P}(\mu)=(\int_\Lambda \lambda_s\,d\mu, \int_\Lambda \lambda_u\,d\mu)$. In view of Corollary 15, for each

$$(\alpha_1,\alpha_2),(\beta_1,\beta_2)\in \operatorname{int}\{\mathscr{P}(\mu)\colon \mu\in\mathscr{M}_\Phi(\Lambda)\}$$

the number $\mathscr{D}(\alpha_1,\alpha_2,\beta_1,\beta_2)$ is given by

$$\frac{1}{\alpha_2}\max\{h_\mu(\Phi)\colon \mathscr{P}(\mu)=(\alpha_1,\alpha_2)\}-\frac{1}{\beta_1}\max\{h_\mu(\Phi)\colon \mathscr{P}(\mu)=(\beta_1,\beta_2)\},$$

where each μ is a measure in $\mathscr{M}_\Phi(\Lambda)$. By Theorem 6 we also have

$$\begin{aligned}\mathscr{D}(\alpha_1,\alpha_2,\beta_1,\beta_2)=&\frac{1}{\alpha_2}\min\{P_\Phi(q_1(-w-\alpha_1)+q_2(v-\alpha_2):(q_1,q_2)\in\mathbb{R}^2\}\\&-\frac{1}{\beta_1}\min\{P_\Phi(q_1(-w-\beta_1)+q_2(v-\beta_2):(q_1,q_2)\in\mathbb{R}^2\}.\end{aligned}$$

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